Randomness and Free Choice Sequences

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In discussions about how to establish liveness properties for distributed protocols, Bob and Mark saw a connection to free choice sequences. They presented these ideas.

Choice sequences. Brouwer in the beginning of his work on Intuitionism limited himself to constructive methods and to so-called lawful sequences. Gradually from 1908 to 1917 he worked out a coherent account of processes generating lawless sequences say of the kind arising from physical processes such as throwing a die. Kleene, Troelstra, and Van Dalen have managed to formalize these ideas another sign that they are coherent. Here are four key axioms as Bob presented them.

Let $\propto, \beta$ be choice sequences.

Axiom 1 (Density) $\forall e : \mathbb{N}\ list. \exists \propto .e \lhd \propto$

For any finite list of numbers e, there is a choice sequence that extends it. (Think of e as presetting a finite number of casts of a die.) This axiom provides *enough* choice sequences.

Axiom 2 (Identity) $\forall \propto, \beta.\ \propto \equiv \beta \vee \propto \not\equiv \beta$

We can decide process identity by its simple name.

Axiom 3 (Equality) Let $x = p$ mean $\forall x.\ \propto (x) = \beta(x)$.

$\forall x, \beta.\ \propto = \beta \vee \propto \neq \beta$

This asserts *independence* of two distinct choice sequences. It essentially says that $\propto \not\equiv \beta$ implies $\propto \neq \beta$.

Axiom 4 (Continuity) $A(\propto) \Rightarrow \exists e : \mathbb{N}\ list.e \lhd \propto \ \&\ \forall \beta.(e \lhd \beta \Rightarrow A(\beta))$

This capture's Brouwer's key insight that any proof about $\propto$ must be based on a finite initial segment of its values.

Fact 1. $\forall \propto \neg \forall y (\propto (y) \neq 0)$

Fact 2. $\neg \forall \propto \exists y (\propto (y) = 0)$. Fact 2 contradicts Markov's Principle.