Justifying the HOL-Nuprl Connection in the Categorical Framework of General Logics

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Motivation

Formal developments often require several man-months of effort and it is highly desirable that their results can be shared between different systems.

$\Rightarrow$ Here formal interoperability provides a *solution*.

The technology is evolving and there will never be a single tool that covers all the needs of formal verification.

$\Rightarrow$ Here formal interoperability is a *requirement*.

Both arguments apply to the connection between HOL and Nuprl.

HOL/Nuprl Overview

HOL

- based on *classical higher-oder logic*
- one of the most popular theorem provers
- a huge amount of mathematics has been developed

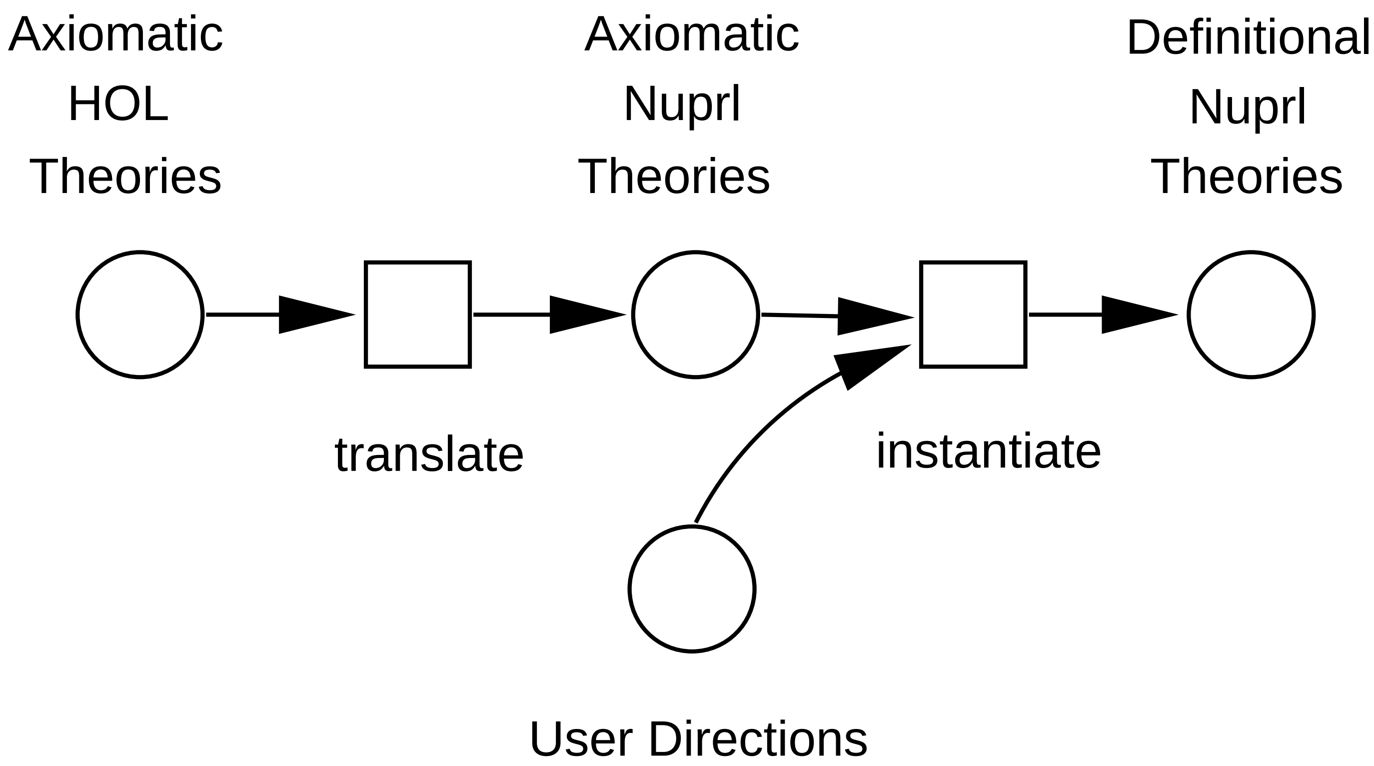
Nuprl

- descendent of Martin Loef's polymorphic type theory
- much more expressive than HOL
- internal *intuitionistic higher-order logic* based on *proposition-as-types* interpretation
- allows program extraction from formal proofs

Previous Work at Bell Labs by Doug Howe

- classical/computational set-theoretic semantics for Nuprl
  (proves that classical Nuprl is consistent)
- exchange format for importing external axiomatic theories into Nuprl
- Nuprl extensions for interactive instantiation of axiomatic theories
- translation from HOL theories into Nuprl theories has been implemented in ML
  (using imperative features)

Translation and Instantiation



Remaining Problems

- translator is not in a form suitable for meta-logical analysis
- correctness of the translation is crucial and should be addressed
- a general methodology to build logic translators
  with theoretical and practical guidelines is missing
- existing implementation is not a separate program
  but integrated into the HOL and Nuprl systems

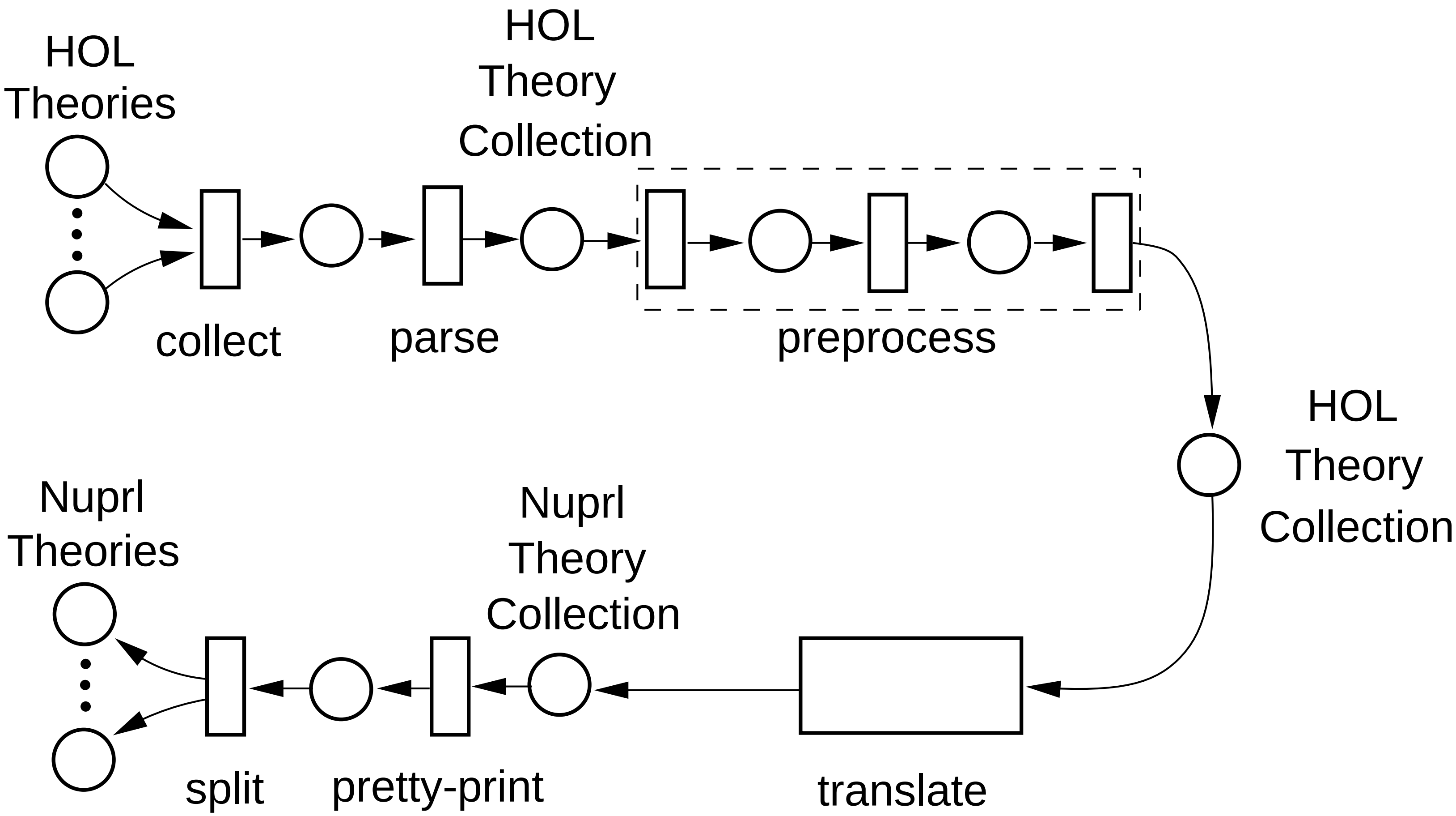
Recent Work at SRI

In order to achieve a better understanding of the translation it has been specified in Maude:

- the specification is *purely equational*,
  i.e. free of side effects and other kinds of imperativity
- the algebraic specification is more *abstract*, more *modular*
  and therefore easier to understand
- the specification is *executable*,
  hence it can be used as a running tool
- the translator has *clear external interfaces*,
  it can be understood independently of the HOL and Nuprl systems

The executable specification has been used to translate all basic HOL theories including theories of arithmetic, sets, lists, trees and a development of combinatory logic.

Architecture of the Translator



Syntactic Categories/Functors of HOL I

Objects	Category
$\Omega \ \triangleright \ \alpha^n : \mathsf{Type}^n \ \triangleright \ \sigma$	**HolFullType**
$\Omega \ \triangleright \ \Delta \ \triangleright \ \alpha^n : \mathsf{Type}^n \ \triangleright \ v^m : \sigma^m \ \triangleright \ a : \sigma$	**HolFullTerm**
$\Omega \ \triangleright \ \Delta \ \triangleright \ \alpha^n : \mathsf{Type}^n \ \triangleright \ v^m : \sigma^m \ \triangleright \ \Gamma \ \Vdash \ t$	**HolFullSen**

Objects	Functor
σ	$\mathit{HolType} : \mathbf{HolFullTypeCtxt} \to \mathbf{Set}$
$a : \sigma$	$\mathit{HolTerm} : \mathbf{HolFullCtxt} \to \mathbf{Set}$
$\alpha^n : \mathsf{Type}^n \ \triangleright \ v^m : \sigma^m \ \triangleright \ \Gamma \ \Vdash \ t$	$\mathit{HolSen} : \mathbf{HolFullSign} \to \mathbf{Cat}$

Syntactic Categories/Functors of HOL II

Objects	Category
Ω	**HolFullTypeSign = HolTypeSign**
$\Omega \triangleright \Delta$	**HolFullSign**
$\Omega \triangleright \alpha^n : \mathsf{Type}^n$	**HolFullTypeCtxt**
$\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m$	**HolFullCtxt**

Objects	Category/Functor
Ω	**HolTypeSign**
Δ	$\mathit{HolSign} : \mathbf{HolFullTypeSign} \to \mathbf{Cat}$
$\alpha^n : \mathsf{Type}^n$	$\mathit{HolTypeCtxt} : \mathbf{HolFullTypeSign} \to \mathbf{Cat}$
$v^m : \sigma^m$	$\mathit{HolCtxt} : \mathbf{HolFullTypeCtxt} \to \mathbf{Cat}$

Syntactic Categories/Functors of Nuprl I

Objects	Category
$\Sigma \ \triangleright \ v^m : \Box^m \ \triangleright \ t$	**NuprlFullTerm**
$\Sigma \ \triangleright \ v^m : S^m \ \triangleright \ t = t' \in T$	**NuprlFullSen**

Objects	Category/Functor
t	$\mathit{NuprlTerm} : \mathbf{NuprlFullCtxt} \to \mathbf{Set}$
$v^m : S^m \ \Vdash \ t = t' \in T$	$\mathit{NuprlSen} : \mathbf{NuprlFullSign} \to \mathbf{Cat}$

Syntactic Categories/Functors of Nuprl II

Objects	Category
Σ	**NuprlFullSign**
$\Sigma \triangleright v^m : \Box^m$	**NuprlFullCtxt**

Objects	Category/Functor
Σ	**NuprlSign**
$v^m : \Box^m$	**NuprlCtxt**
$v^m : S^m$	**NuprlTypedCtxt**
$v^m : S^m$	$\mathit{NuprlTypedCtxt} : \mathbf{NuprlFullSign} \to \mathbf{Cat}$
$v^m : S^m$	$\mathit{NuprlTypedCtxt} : \mathbf{NuprlFullCtxt} \to \mathbf{Cat}$

Overview of the Translation

The HOL-Nuprl translation can be specified by a functor

$$\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlTh_0}$$

together with a natural transformation

$$\alpha : HolSen \to NuprlSen \circ \Phi$$

where

$$HolSen : \mathbf{HolFullSign} \to \mathbf{Cat},$$

$$NuprlSen : \mathbf{NuprlTh_0} \to \mathbf{Cat}.$$

Translating Full Signatures

First we define the functor

$$\Phi : \mathbf{HolFullSign} \to \mathbf{NurplTh_0}$$

$$\Phi(\Sigma) = (\Phi(\Sigma), \Psi(\Sigma))$$

with

$$\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlFullSign}$$

$$\Psi : \prod \Sigma : \mathbf{HolFullSign} \;.\; NuprlAxSet(\Phi(\Sigma))$$

where

$$NuprlAxSet : \mathbf{NuprlFullSign} \to \mathbf{Cat}$$

Translating Type Signatures

$$\Phi : \mathbf{HolFullTypeSign} \to \mathbf{NuprlFullSign}$$

$$\Psi : \prod \Omega : \mathbf{HolFullTypeSign}\ .\ \mathit{NuprlAxSet}(\Phi(\Omega))$$

where

$$\mathit{NuprlAxSet} : \mathbf{NuprlFullSign} \to \mathbf{Cat}$$

1. $\Phi(\{\nu : \mathsf{Type}^n{\to}\mathsf{Type}\}) = \{\nu : \Box^n{\to}\Box\}$,
2. $\Phi(\emptyset) = \emptyset$,
3. $\Phi(\Omega_1 \cup \Omega_2) = \Phi(\Omega_1) \cup \Phi(\Omega_2)$
4. $\Psi(\{\nu : \mathsf{Type}^n{\to}\mathsf{Type}\}) = \{\alpha, \ldots, \alpha_n : \mathsf{S} \ \Vdash\ \nu(\alpha, \ldots, \alpha_n) \in \mathsf{S}\}$,
5. $\Psi(\emptyset) = \emptyset$,
6. $\Psi(\Omega_1 \cup \Omega_2) = \Psi(\Omega_1) \cup \Psi(\Omega_2)$

with

$$\mathsf{S} = \{T : \mathsf{U}_1 \mid \exists x : T\ .\ \mathsf{True}\},$$

Translating Types

$$\alpha : \mathit{HolType} \to \mathit{NuprlTerm} \circ \Phi$$

where

$$\mathit{HolType} : \mathbf{HolFullTypeCtxt} \to \mathbf{Set}$$

$$\Phi : \mathbf{HolFullTypeCtxt} \to \mathbf{NuprlFullCtxt}$$

$$\mathit{NuprlTerm} : \mathbf{NuprlFullCtxt} \to \mathbf{Set}$$

1. $\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\alpha_i^n) = \alpha_i^n$,

2. $\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\mathsf{o}) = \mathsf{Bool}$,

3. $\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\sigma \to \tau) =$
   $\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\sigma) \to \alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\tau)$,

4. $\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\nu(\sigma_1, \ldots, \sigma_n)) =$
   $\nu(\alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\sigma_1), \ldots, \alpha(\Omega \, \triangleright \, \alpha^n : \mathsf{Type}^n)(\sigma_n))$

Translating Signatures

$$\alpha : \mathit{HolSign} \to \mathbf{NuprlSign}$$

$$\beta(\Omega) : \prod \Delta : \mathit{HolSign}(\Omega) \,.\, \mathit{NuprlAxSet}(\alpha(\Omega)(\Delta))$$

where

$$\mathit{HolSign} : \mathbf{HolTypeSign} \to \mathbf{Cat}$$

$$\mathbf{NuprlSign} : \mathbf{HolTypeSign} \to \mathbf{Cat}$$

$$\mathit{NuprlAxSet} : \mathbf{NuprlFullSign} \to \mathbf{Cat}$$

1. $\alpha(\Omega)(\{\alpha^n : \mathsf{Type}^n \;\triangleright\; c : \sigma\}) = \{c : \Box^n \to \Box\}$,
2. $\alpha(\Omega)(\emptyset) = \emptyset$,
3. $\alpha(\Omega)(\Delta_1 \cup \Delta_2) = \alpha(\Omega)(\Delta_1) \cup \alpha(\Omega)(\Delta_2)$
4. $\beta(\Omega)(\{\alpha^n : \mathsf{Type}^n \;\triangleright\; c : \sigma\}) =$
   $\{\alpha^n : \mathsf{S}^n \;\Vdash\; c(\alpha^n) \in \alpha(\Omega \;\triangleright\; \alpha^n : \mathsf{Type}^n)(\sigma)\}$,
5. $\beta(\Omega)(\emptyset) = \emptyset$,
6. $\beta(\Omega)(\Delta_1 \cup \Delta_2) = \beta(\Omega)(\Delta_1) \cup \beta(\Omega)(\Delta_2)$

Translating Terms

$$\alpha : \mathit{HolTerm} \to \mathit{NuprlTerm} \circ \Phi$$

where

$$\mathit{HolTerm} : \mathbf{HolFullCtxt} \to \mathbf{Set}$$

$$\Phi : \mathbf{HolFullCtxt} \to \mathbf{NuprlFullCtxt}$$

$$\mathit{NuprlTerm} : \mathbf{NuprlFullCtxt} \to \mathbf{Set}$$

1. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(v_i^m) = v_i^m$,

2. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(= : \tau \to \tau \to \mathsf{o}) = (=_b)(\alpha(\Omega)(\tau))$,

3. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(\Rightarrow : \mathsf{o} \to \mathsf{o} \to \mathsf{o}) = (\Rightarrow_b)$,

4. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(c : \tau[\tau^k / \beta^k]) =$
   $c(\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\tau_1^n), \ldots, \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\tau_n^n))$,
   where $(\beta^k : \mathsf{Type}^k \triangleright c : \tau) \in \Delta$ and we have types $\alpha^n : \mathsf{Type}^n \triangleright \tau_i^n$

5. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(((a : \sigma \to \tau)(b : \sigma)) : \tau) =$
   $(\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(a : \sigma \to \tau)$
   $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)(b : \sigma))$,

6. $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m)((\lambda\, v : \sigma \,.\, (a : \tau)) : \sigma \to \tau) =$
   $(\lambda\, v \,.\, \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright v^m : \sigma^m v : \sigma)(a : \tau))$

with

- $\Rightarrow_b = \lambda\ x, y$. if x then y else tt,
- $=_b(\rho) = \lambda\ x, y\ .\ \downarrow_b (x = y \in \rho)$.

Translating Sentences

$$\alpha : HolSen \to NuprlSen \circ \Phi$$

where

$$HolSen : \mathbf{HolFullSign} \to \mathbf{Cat}$$

$$NuprlSen : \mathbf{NuprlFullSign} \to \mathbf{Cat}$$

$$\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlFullSign}$$

$$\alpha(\Omega \; \triangleright \; \Delta)(\alpha^n : \mathsf{Type}^n \; \triangleright \; v^m : \sigma^m \; \triangleright \; \Gamma \; \Vdash \; t) =$$

$$\Phi(\alpha^n : \mathsf{Type}^n)$$

$$\alpha(\Omega \; \triangleright \; \alpha^n : \mathsf{Type}^n)(v^m : \sigma^m)$$

$$\triangleright$$

$$\alpha(\Omega \; \triangleright \; \Delta \; \triangleright \; \alpha^n : \mathsf{Type}^n \; \triangleright \; v^m : \sigma^m)(\Gamma)$$

$$\Vdash$$

$$\mathsf{axiom} \in \uparrow \alpha(\Omega \; \triangleright \; \Delta \; \triangleright \; \alpha^n : \mathsf{Type}^n \; \triangleright \; v^m : \sigma^m)(t)$$

with

1. $\alpha(\Omega \,\triangleright\, \Delta \,\triangleright\, \alpha^n : \mathsf{Type}^n \,\triangleright\, v^m : \sigma^m)([]) = []$

2. $\alpha(\Omega \,\triangleright\, \Delta \,\triangleright\, \alpha^n : \mathsf{Type}^n \,\triangleright\, v^m : \sigma^m)([t_1, \ldots, t_n]) =$
   $[\ h_1\ :\uparrow \alpha(\Omega \,\triangleright\, \Delta \,\triangleright\, \alpha^n : \mathsf{Type}^n \,\triangleright\, v^m : \sigma^m)(t_1), \ldots,$
   $h_n\ :\uparrow \alpha(\Omega \,\triangleright\, \Delta \,\triangleright\, \alpha^n : \mathsf{Type}^n \,\triangleright\, v^m : \sigma^m)(t_n)\ ]$

Correctness of the Translation

The HOL-Nuprl translation satisfies a syntactical correctness condition, namely that it constitutes a

map between entailment systems

of HOL and Nuprl in the sense of general logics.

Apart from categorical well-formedness the essential condition to be verified is

$$\Gamma \vdash_{Hol} \phi \text{ implies } \alpha_\Sigma(\Gamma) \vdash_{ClassicalNuprl} \alpha_\Sigma(\phi)$$
$$\text{for each full HOL signature } \Sigma$$

$\Gamma \vdash \phi$ means that the sentence ϕ can be *derived (by a sequential proof)* using the sentences in Γ as hypotheses.

Sequential proofs are defined by induction using *instances of inference rules*.

Hence to verify the condition above it is sufficient to show that each translated instance of an HOL-inference rule can be derived in Classical Nuprl.

HOL-Rules I

$$\frac{}{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ t \ \Vdash\ t} \qquad \text{(ASSUME)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H \ \Vdash\ t}{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H' \ \Vdash\ t} \ \text{if } H \text{ and } H' \text{ are equal as sets} \qquad \text{(STRUCT)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H_2 \ \Vdash\ t_1 \qquad \alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H_1 \ \Vdash\ t_1 \Rightarrow t_2}{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H_1, H_2 \ \Vdash\ t_2} \qquad \text{(MP)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H \ \Vdash\ t_2}{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \ \triangleright\ H - \{t_1\} \ \Vdash\ t_1 \Rightarrow t_2} \qquad \text{(DISCH)}$$

$$\frac{}{\alpha^n : \mathsf{Type}^n \ \triangleright\ v^m : \sigma^m \triangleright\ \Vdash\ t = t} \qquad \text{(REFL)}$$

HOL-Rules II

$$\frac{}{\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \triangleright \;\Vdash\; ((\lambda\, v : \sigma\, .\, s)\, t) = s[t/v]} \quad \text{(BETA\_CONV)}$$

$$\frac{\begin{array}{c}\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H_i^k \;\Vdash\; t_i^k = t_i'^k \text{ for all } i \\ \alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H \;\Vdash\; t[t^k/z]\end{array}}{\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H^k, H \;\Vdash\; t[t'^k/z]} \quad \text{(SUBST)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H \;\Vdash\; s = t}{\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H \;\Vdash\; (\lambda\, v : \sigma\, .\, s) = (\lambda\, v : \sigma\, .\, t)} \quad \text{(ABS)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \;\triangleright\; v^m : \sigma^m \;\triangleright\; H \;\Vdash\; t}{\beta^k : \mathsf{Type}^k \;\triangleright\; (v^m : \sigma^m \;\triangleright\; H \;\Vdash\; t)[\tau^n/\alpha^n]} \quad \text{(INST\_TYPE)}$$

provided that we have types $\beta^k : \mathsf{Type}^k \;\triangleright\; \tau_i^n$ and the α_i^n do not occur in H.

Remaining Questions/Future Work

- What alternative translations are conceivable and
  how can they explained in the framework of general logics ?

- What are the right categorical methods/tools
  to deal with logics and type theories on a syntactical level ?

- What is the role of meta logics like ISABELLE, MEL, MetaPRL in this context ?

- Is there a need/use of a semantic connection (e.g. in the sense of institutions) ?

- Is there a categorical simplification/generalization of Allen's Semantics ?

- Explore other connections within the family of
  higher-order logic theorem provers (Nuprl, HOL, PVS, COQ, LEGO)

- Apply the methodology to develop further translators
  which can be directly used as valuable tools by formal method researchers.