Intersections, Unions and Games

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Understanding Girard's Ludics 'bottom-up'

Partial Proofs Game

- In Nuprl type theory we use 'propositions as types of proofs'
- Girard uses 'propositions as sets of "paraproofs"'
- "Paraproofs" are partial proofs in a restricted sequent calculus ("Hypersequent Calculus")

Hypersequent Calculus - Syntax

- Based on Linear Logic

$\perp$ $\otimes$

Hyper equent Calculu - Rule

Positive Rules

$$\frac{P_{i,1} \vdash \Sigma_1 \quad \ldots \quad P_{i,k_i} \vdash \Sigma_{k_i}}{\vdash P_{1,1}^\perp \otimes \ldots \otimes P_{1,k_1}^\perp) \oplus \ldots \oplus (P_{n,1}^\perp \otimes \ldots \otimes P_{n,k_n}^\perp), \Sigma_1, \ldots, \Sigma_{k_i}} \ (\ ) \qquad \frac{}{\vdash \Sigma} \ (\ ')$$

Neg tive Rules

$$\frac{\vdash P_{1,1}, \ldots, P_{1,k_1}, \Sigma \quad \ldots \quad \vdash P_{n,1}, \ldots, P_{n,k_n}, \Sigma}{(P_{1,1}^\perp \otimes \ldots \otimes P_{1,k_1}^\perp) \oplus \ldots \oplus (P_{n,1}^\perp \otimes \ldots \otimes P_{n,k_n}^\perp) \vdash \Sigma} \ (\mathrm{L}) \qquad \frac{}{\perp \vdash}$$

- (R) and (L) are normal rules, (R') is a "paralogism" and (L') is a paralogism iff Σ contains anything but $\perp$s.
- Anything that uses (R), (L), (R') and (L') is a "paraproof". Proof is a paraproof that does not use paralogisms.
- In p

Cut and Cut Elimination

- Girard gives "eager" cut elimination algorithm that converts paraproofs to paraproofs and proof(T(to)TjT(p)TjT(ru)-(which)T(s)T(no)T a jon paralogism.

The Game

- The game is played ween a play who tries to construct a parap of $P \vdash$ and player who tries to construct a parap of P (which are combined into a cut of empty sequent).
onstructed "onroT(demand")Tjasnecessarytoproceed

Game Example

Consider a game when $P = (P_1^\perp \otimes$

Strategie

- Each paraproof defines a complete strategy.
- And each deterministic strategy is a paraproof.
- Winning strategy = proof!