Adding Bar Induction to Nuprl

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Bar Induction in Nuprl

- Bar Induction is true in both classical and intuitionistic logic
- It proves Fan, which has useful consequences
- To add Bar Induction to Nuprl we need its realizer
- We will define and prove the realizer correct using a weak bar induction rule.
- This seems to be a general "bootstrapping" method.

Warm-up: Induction on $\mathbb{N}$

$$\forall[A\colon \mathbb{N} \to \mathbb{P}].\ A(0) \Rightarrow (\forall n\colon \mathbb{N}.\ A(n) \Rightarrow A(n+1)) \Rightarrow \forall n\colon \mathbb{N}.\ A(n)$$

The realizer must have the form: $\lambda b.\lambda s.\lambda n.F(b, s, n)$ where

$$\begin{array}{l} A\colon \mathbb{N} \to \mathbb{P} \\ b\colon A(0) \\ s\colon n\colon \mathbb{N} \to A(n) \to A(n+1) \\ n\colon \mathbb{N} \\ \vdash F(b, s, n) \in A(n) \end{array}$$

Let's prove this by induction on $n \in \mathbb{N}$ and define $F(b, s, n)$ as we go.

Proof of realizer for Induction on $\mathbb{N}$

Inductive proof has two subgoals:

$$\begin{array}{ll} A\colon \mathbb{N} \to \mathbb{P} \\ b\colon A(0) \\ s\colon n\colon \mathbb{N} \to A(n) \to A(n+1) \\ \vdash F(b,s,0) \in A(0) \end{array}$$

$$\begin{array}{ll} A\colon \mathbb{N} \to \mathbb{P} \\ b\colon A(0) \\ s\colon n\colon \mathbb{N} \to A(n) \to A(n+1) \\ n\colon \mathbb{N} \\ F(b,s,n) \in A(n) \\ \vdash F(b,s,n+1) \in A(n+1) \end{array}$$

So we define

$$F(b,s,n) = \textbf{if } n = 0 \textbf{ then } b \textbf{ else } s(n-1, F(b,s,n-1))$$

(otherwise known as primitive recursion)

Weak induction rule on $\mathbb{N}$

By giving its realizer, we have proved the induction principle on $\mathbb{N}$.
But we use induction on $\mathbb{N}$ in the proof!
But we needed only this weaker rule:

$$\frac{H \vdash G(0) \in P(0) \qquad H, n\colon \mathbb{N}, G(n) \in P(n) \vdash G(n+1) \in P(n+1)}{H, n\colon \mathbb{N} \vdash G(n) \in P(n)}$$

Here the conclusions are "typing judgements" rather than arbitrary propositions.
So we "bootstrapped" from the weaker induction rule to the full induction principle.
Next: Bar Induction

Bar Induction

Notation: $T\colon Type$, $t\colon T$, $l\colon T$ **list**, $n\colon \mathbb{N}$, $f : \mathbb{N} \to T$

$$
\begin{array}{lcl}
l \cdot t & = & l \oplus [t] \quad (\oplus \text{ is append}) \\
\mathbf{upto}(n) & = & [0, 1, \ldots, n-1] \\
\overline{f}(n) & = & \mathbf{map}(f, \mathbf{upto}(n))
\end{array}
$$

The Bar Induction principle is:

$$
\begin{array}{l}
\forall [T\colon Type].\ \forall [A, B\colon T\ \mathbf{list} \to \mathbb{P}]. \\
(\forall l\colon T\ \mathbf{list}.\ B(l) \vee \neg B(l)) \Rightarrow \\
(\forall l\colon T\ \mathbf{list}.\ B(l) \Rightarrow A(l)) \Rightarrow \\
(\forall l\colon T\ \mathbf{list}.\ (\forall t\colon T.\ A(l \cdot t)) \Rightarrow A(l)) \Rightarrow \\
(\forall f : \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(\overline{f}(n))) \Rightarrow \\
A(\mathbf{nil})
\end{array}
$$

Classical equivalent of Bar Induction

Leaving out $(\forall l\colon T\ \textbf{list}.\ B(l) \vee \neg B(l))$ and taking contrapositives:

$$\begin{array}{l}
(\forall l\colon T\ \textbf{list}.\ \neg A(l) \Rightarrow \neg B(l)) \Rightarrow \\
(\forall l\colon T\ \textbf{list}.\ \neg A(l) \Rightarrow (\exists t\colon T.\ \neg A(l \cdot t))) \Rightarrow \\
\neg A(\textbf{nil}) \Rightarrow \\
\exists f : \mathbb{N} \to T.\ \forall n\colon \mathbb{N}.\ \neg B(\bar{f}(n))
\end{array}$$

Which follows from:

$$\begin{array}{l}
\neg A(\textbf{nil}) \Rightarrow \\
(\forall l\colon T\ \textbf{list}.\ \neg A(l) \Rightarrow (\exists t\colon T.\ \neg A(l \cdot t))) \Rightarrow \\
\exists f : \mathbb{N} \to T.\ \forall n\colon \mathbb{N}.\ \neg A(\bar{f}(n))
\end{array}$$

Which is clearly true.

Realizer for Bar Induction

First, generalize slightly:

$$\forall[T\colon \mathit{Type}].\ \forall[A, B\colon T\ \textbf{list} \to \mathbb{P}].$$
$$(\forall l\colon T\ \textbf{list}.\ B(l) \vee \neg B(l)) \Rightarrow$$
$$(\forall l\colon T\ \textbf{list}.\ B(l) \Rightarrow A(l)) \Rightarrow$$
$$(\forall l\colon T\ \textbf{list}.\ (\forall t\colon T.\ A(l \cdot t)) \Rightarrow A(l)) \Rightarrow$$
$$\forall l\colon T\ \textbf{list}.\ [\forall f\colon \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(l \oplus \overline{f}(n))] \Rightarrow$$
$$A(l)$$

The realizer must have the form: $\lambda d.\lambda b.\lambda s.\lambda l.BR(d, b, s, l)$ where

$$d\colon (l\colon T\ \textbf{list} \to B(l) + \neg B(l))$$
$$b\colon (l\colon T\ \textbf{list} \to B(l) \to A(l))$$
$$s\colon (l\colon T\ \textbf{list} \to (t\colon T \to A(l \cdot t)) \to A(l))$$
$$l\colon T\ \textbf{list}$$
$$\forall f\colon \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(l \oplus \overline{f}(n))$$
$$\vdash BR(d, b, s, l \oplus \textbf{nil}) \in A(l \oplus \textbf{nil})$$

Realizer for Bar Induction, cont'd

$$
\begin{array}{l}
d\colon (l\colon T\ \mathbf{list} \to B(l) + \neg B(l)) \\
b\colon (l\colon T\ \mathbf{list} \to B(l) \to A(l)) \\
s\colon (l\colon T\ \mathbf{list} \to (t\colon T \to A(l \cdot t)) \to A(l)) \\
x\colon T\ \mathbf{list} \\
\forall f : \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(x \oplus \bar{f}(n)) \\
\vdash BR(d, b, s, x \oplus \mathbf{nil}) \in A(x \oplus \mathbf{nil})
\end{array}
$$

With the following definition

$$
\begin{array}{rl}
BR(d, b, s, l) = & \mathbf{case}(d(l)) \\
 & \mathbf{inl}(p)\colon\ b(l, p) \\
 & \mathbf{inr}(\_)\colon\ s(l, \lambda t.BR(d, b, s, l \cdot t))
\end{array}
$$

we prove it by Bar Induction(!) using $\lambda l.B(x \oplus l)$ for the bar.

Proof of Realizer for Bar Induction

Given

$$
\begin{aligned}
&d\colon (\forall l\colon\ T\ \textbf{list}.\ B(l) \vee \neg B(l))\\
&b\colon (\forall l\colon\ T\ \textbf{list}.\ B(l) \Rightarrow A(l))\\
&s\colon (\forall l\colon\ T\ \textbf{list}.\ (\forall t\colon\ T.\ A(l \cdot t)) \Rightarrow A(l))\\
&x\colon\ T\ \textbf{list}\\
&\forall f : \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(x \oplus \bar{f}(n))
\end{aligned}
$$

We have to prove four things. Two of them say that $\lambda l.B(x \oplus l)$ is a bar:

$$\forall l\colon\ T\ \textbf{list}.\ B(x \oplus l) \vee \neg B(x \oplus l)$$

$$\forall f : \mathbb{N} \to T.\ \exists n\colon \mathbb{N}.\ B(x \oplus \bar{f}(n))$$

(these are immediate from our hypotheses)

Proof of Realizer for Bar Induction, cont'd

$$d\colon (\forall l\colon T\ \textbf{list}.\ B(l) \vee \neg B(l))$$
$$b\colon (\forall l\colon T\ \textbf{list}.\ B(l) \Rightarrow A(l))$$
$$x\colon T\ \textbf{list}$$

The base case is:

$$\forall l\colon T\ \textbf{list}.\ B(x \oplus l) \Rightarrow BR(d, b, s, x \oplus l) \in A(x \oplus l)$$

So we need

$$l\colon T\ \textbf{list}$$
$$p\colon B(x \oplus l)$$
$$\vdash BR(d, b, s, x \oplus l) \in A(x \oplus l)$$

In this case, $BR(d, b, s, x \oplus l) = b(x \oplus l, p) \in A(x \oplus l)$

Proof of Realizer for Bar Induction, cont'd

The induction step is:

$$\begin{array}{l} d\colon (\forall l\colon T\ \textbf{list}.\ B(l) \vee \neg B(l)) \\ s\colon (\forall l\colon T\ \textbf{list}.\ (\forall t\colon T.\ A(l \cdot t)) \Rightarrow A(l)) \\ l\colon T\ \textbf{list} \\ \forall t\colon T.\ BR(d, b, s, x \oplus (l \cdot t)) \in A(x \oplus (l \cdot t)) \\ \vdash BR(d, b, s, x \oplus l) \in A(x \oplus l) \end{array}$$

If $d(x \oplus l) = \textbf{inl}(p)$ then we are in the base case. Otherwise, $d(x \oplus l) = \textbf{inr}(\_)$ and

$$BR(d, b, s, x \oplus l) = s(x \oplus l, \lambda t.BR(d, b, s, (x \oplus l) \cdot t))$$

So, since $x \oplus (l \cdot t) = (x \oplus l) \cdot t$,

$$BR(d, b, s, x \oplus l) \in A(x \oplus l)$$

QED.

Bar induction rule in Nuprl

We defined the bar-recursion BR and showed that it realizes the general Bar Induction principle. In the proof we used only this weaker bar induction rule:

$$\frac{\begin{array}{c} H \vdash T \in Type \\ H, l\colon T\ \textbf{list} \vdash B(l) \vee \neg B(l) \qquad H, f : \mathbb{N} \to T \vdash \exists n\colon \mathbb{N}.\ B(\bar{f}(n)) \\ H, l\colon T\ \textbf{list}, B(l) \vdash G(l) \in P(l) \\ H, l\colon T\ \textbf{list}, \forall t\colon T.\ G(l \cdot t) \in P(l \cdot t) \vdash G(l) \in P(l) \end{array}}{H \vdash G(\textbf{nil}) \in P(\textbf{nil})}$$

We added this rule to Nuprl and used it to prove the general Bar Induction (with realizer BR).

Realizer for Fan

$$
\begin{aligned}
&dbar(B) = (\forall l\colon \mathbb{B}\ \textbf{list}.\ B(l) \vee \neg B(l)) \wedge (\forall f : \mathbb{N} \to \mathbb{B}.\ \exists n\colon \mathbb{N}.\ B(\bar{f}(n))) \\
&ubar(B) = \exists k\colon \mathbb{N}.\ \forall f : \mathbb{N} \to \mathbb{B}.\ \exists n\colon \mathbb{N}.\ n < k \wedge B(\bar{f}(n)) \\
&Fan = \forall B\colon \mathbb{B}\ \textbf{list} \to \mathbb{P}.\ dbar(B) \Rightarrow ubar(B)
\end{aligned}
$$

Once we have Bar Induction, we can prove the Fan theorem by bar induction. We did the proof in Nuprl and extracted the following realizer:

$$
\begin{aligned}
\lambda B, \langle d, \_\rangle.\ BR\quad & (d, \\
& \lambda \_, b.\ \langle 1, \lambda \_.\ \langle 0, b\rangle\rangle, \\
& \lambda \_, e.\ \textbf{let}\ k_1, left = e(\textbf{false})\ \textbf{in} \\
& \qquad \textbf{let}\ k_2, right = e(\textbf{true})\ \textbf{in} \\
& \qquad \langle 1 + \textbf{max}(k_1, k_2), \\
& \qquad \lambda f.\ \textbf{let}\ n_1, br = right(\lambda n.\ f(n+1))\ \textbf{in} \\
& \qquad\quad \textbf{let}\ n_2, bl = left(\lambda n.\ f(n+1))\ \textbf{in} \\
& \qquad\quad \textbf{if}\ f(0)\ \textbf{then}\ \langle n_1 + 1, br\rangle \textbf{else}\ \langle n_2 + 1, bl\rangle \\
& \textbf{nil})
\end{aligned}
$$