Order-theoretic Differences Between Two Variants of Type Theory

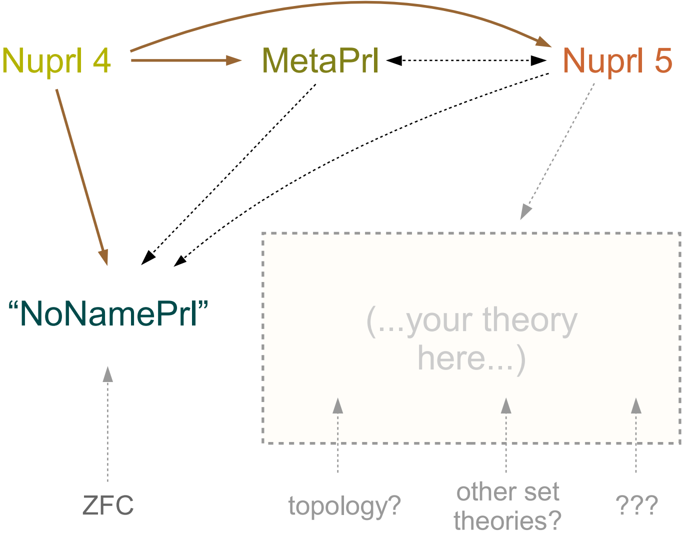
photo courtesy philip greenspun

$$\bigcap_{T:\mathbb{U}_1} T \to T$$

any nontrivial subtypes?

(other than itself and the empty type)

photo courtesy philip greenspun



Nuprl 4
MetaPrl
Nuprl 5
"NoNamePrl"
(...your theory here...)
ZFC
topology?
other set theories?
???

$$\{t\}^\flat \triangleq \{t\}_{\mathsf{Base}}$$

$$\{t\}^\flat \subseteq T \quad \text{iff} \quad t \in T$$

$$\circlearrowright \triangleq (\lambda x.x\,x)(\lambda x.x\,x)$$

$$\mathsf{inl}(3) \div (\mathbb{N} \to \mathbb{N})$$

$$\mathsf{inl}(3) \div (\mathbb{N} \to \mathbb{N}) \in \{\circlearrowright\}^\flat$$

(and also Top, $\{\circlearrowright\}^\flat \cup \{0\}^\flat$, ...)

$$\{\lambda x.x\}^\flat \subseteq \bigcap_{T:\mathbb{U}_1} T \to T \subseteq \bigcap_{x:\mathsf{Base}} \{x\}^\flat \to \{x\}^\flat$$

$$\bigcap \left\{ \begin{array}{rcl} \{0\}^\flat & \to & \{0\}^\flat \\ \{1\}^\flat & \to & \{1\}^\flat \\ \{\lambda x.x+3\}^\flat & \to & \{\lambda x.x+3\}^\flat \\ \{\mathsf{inl}(3) \div (\mathbb{N} \to \mathbb{N})\}^\flat & \to & \{\mathsf{inl}(3) \div (\mathbb{N} \to \mathbb{N})\}^\flat \\ \{\mathbb{U}_1\}^\flat & \to & \{\mathbb{U}_1\}^\flat \\ \{\mathbb{U}_{337}\}^\flat & \to & \{\mathbb{U}_{337}\}^\flat \\ \{\mathbb{U}_{2038}\}^\flat & \to & \{\mathbb{U}_{2038}\}^\flat \\ & \vdots & \end{array} \right.$$

$\mathbb{U}_1 \in \mathsf{Base}$
$\mathsf{Base} \in \mathbb{U}_1$

for every closed term t,

$$f\,t \in \{t\}^\flat \qquad \text{since} \qquad f \in \{t\}^\flat \to \{t\}^\flat$$

i.e.,

$$f\,t \sim t \sim (\lambda x.x)\,t$$

hence,

$$f \sim \lambda x.x$$

radically constructive	ebulliently classical

pure collection ability

$\subseteq$

$\succcurlyeq$

$$0, 1, 2, \lambda x.x, \mathsf{inl}(\cdot), \mathsf{inr}(\cdot), \langle \cdot, \cdot \rangle, \mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3,$$
$$\lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x)),$$
$$\ldots$$

$\prod_{T:\mathbb{U}_i} T + \neg T$
extensional type equality
respect for $\succcurlyeq$
bellwethers
separativeness
singletons built from well-founded trees

very tight connection to mainstream set theory only the behavior **absolutely required** by

$$\bigcap_{T:\mathbb{U}_1} T \to T$$

in pseudocode...

$$\lambda x.\mathsf{if}\ x \in \tilde{U}_1\ \mathsf{then}\ x\ \mathsf{else}\ \circlearrowright$$

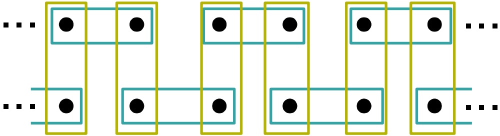
where

$$\tilde{U}_1 \triangleq \bigcup_{\substack{T:\mathbb{U}_1 \\ T \subsetneq \mathsf{Top}}} T$$

T_1 and T_2 both infinite, but…

$$T_1 \cup T_2$$

…is a singleton.



actually works. . .

$$\lambda x. \quad \mathsf{case}\ xm(\{x\}_{\tilde{U}_1})\ \mathsf{of} \\ \mathsf{inl}(\_) \mapsto x \mid \\ \mathsf{inr}(\_) \mapsto \circlearrowright$$

for any $xm \in \prod_{T:\mathbb{U}_2} T + \neg T$.

$$\Gamma, f : \prod_{T:\mathbb{U}_2} T + \neg T, \Delta \vdash C$$

mixing notation a bit…

if $T \in T' \subsetneq \mathsf{Top}$ then $\mathsf{rank}(T) < \mathsf{rank}(T')$,
where rank is conveniently well-ordered.

$\mathbb{U}_1 \in \tilde{U}_1$ would mean that for some T…

$$\begin{array}{ccc} \mathbb{U}_1 & \in & T \\ T & \in & \mathbb{U}_1 \end{array}$$

whence

$$\mathsf{rank}(\mathbb{U}_1) < \mathsf{rank}(T) < \mathsf{rank}(\mathbb{U}_1)$$

$$(\bigcap_{T:\mathbb{U}_1} T \to T) \quad \cap \quad (\{\mathbb{U}_1\}^\flat \to \{0\}^\flat)$$

$$\cap\!\!\!\!\!-^{\bullet}$$

$$(\bigcap_{T:\mathbb{U}_1} T \to T) \quad \cap \quad (\{\mathbb{U}_1\}^\flat \to \{1\}^\flat)$$

where $\{\mathbb{U}_1\}^\flat \triangleq \{\mathbb{U}_1\}_{\mathbb{U}_2}$, $\{0\}^\flat \triangleq \{0\}_{\mathbb{N}}$, etc.

$$\lambda x. \quad \mathsf{case}\ xm(\{x\}_{\tilde{U}_1^+} \not\equiv \{\mathbb{U}_1\}_{\tilde{U}_1^+})\ \mathsf{of}$$
$$\mathsf{inl}(\_) \mapsto x \mid$$
$$\mathsf{inr}(\_) \mapsto 0$$

where $\tilde{U}_1^+ \triangleq \tilde{U}_1 \cup \{\mathbb{U}_1\}^\flat$

relies on $\{\mathbb{U}_1\}^\flat \Cap^{\cdot} \tilde{U}_1$,
which follows from minimality

Fin.

image acknowledgments

the small scenes of fall foliage were both cropped from the same photograph, copyright 1994 by Philip Greenspun…

http://philip.greenspun.com/images/pcd0737/saranac-lake-15 rough side notes...

in general, these arguments happen within a classical metatheory (ZFC plus some inaccessibles).

for convenience, term variables like t and T are typically implicitly assumed to range over closed terms.

p. 2

"other than itself and the empty type" is modulo $\equiv$.

p. 4

$\triangleq$ introduces a definition.

the flat notation is used here to save space (and admittedly also for aesthetic reasons).

$\{t\}^\flat$ has no nontrivial subtypes. [$t' \in T \subseteq \{t\}^\flat$ implies $t' \sim t$, whence $t \in T$, whence $\{t\}^\flat \subseteq T$.]

pp. 6–7

this is a tweak of Mark's proof. the sandwiching argument using $\bigcap_{x:\mathsf{Base}} \dots$ is not strictly necessary; I just wanted to emphasize a particular way of thinking about what is fundamentally going on here.

the big intersection in the middle of p. 6 portrays $\bigcap_{x:\mathsf{Base}} \{x\}^\flat \to \{x\}^\flat$ as an infinite tree of singletons, with each branch exactly specifying (up to $\sim$) the output of a single input. this kind of thing also happens to be is essential to Doug's framework (although of course not using Base).

the inclusions at the top of p. 6 are trivial, so the main direction of the argument is $\bigcap_{x:\mathsf{Base}} \{x\}^\flat \to \{x\}^\flat \subseteq \{\lambda x.x\}^\flat$.

the conclusion can be generalized to $\{\lambda x.x\}^\flat \equiv \bigcap_{T:\mathbb{U}_1} T \to T \equiv \bigcap_{T:\mathbb{U}_2} T \to T \equiv \bigcap_{T:\mathbb{U}_3} T \to T \equiv \cdots$.

p. 4–7

in the orange pages, the object theory is the Nuprl 5 type theory.

in subsequent pages, the object theory is the second variant of type theory: the one arising from Howe's classical, set-theoretic semantics.

the answer to the question posed on p. 2 is "no" for the first variant but "yes" (dramatically yes) for the second.

p. 10

$\succsim$ is the full operational preorder. it's written with a squiggly symbol here to emphasize its relationship to $\sim$.

respect for $\succsim$ means that $t' \succsim t$ and $t \in T$ imply $t' \in T$.

a bellwether of a type S is a term $t \in S$ such that $t' \succsim t$ for every $t' \in S$. consequently, $S \subseteq T$ iff $t \in T$.

p. 11

the λ term is pseudocode in the sense that there is no corresponding if-then-else built into the pure computation system—no primitive that allows you to branch based on membership in a general type.

what we're looking for is a term $f_1 \in \bigcap_{T:\mathbb{U}_1} T \to T$ such that $f_1\, t_\circ \sim \circlearrowright$ for every closed term $t_\circ \notin \tilde{U}_1$.

it isn't obvious that such a $t_\circ$ even exists, but, if it did, this would imply

$$f_1 \not\sim \lambda x.x,$$

which would already be a marked difference from what we saw in the orange part of the talk (with the Nuprl 5 object theory).

p. 11 (cont'd)

if, moreover, $t_\circ \in T_\circ \subsetneq \mathsf{Top}$ for some $T_\circ \in \mathbb{U}_\ell$ and some $\ell > 1$, then

$$f_1 \notin \{t_\circ\}_{T_\circ} \to \{t_\circ\}_{T_\circ} \supseteq \bigcap_{T:\mathbb{U}_\ell} T \to T,$$

whence

$$f_1 \notin \bigcap_{T:\mathbb{U}_\ell} T \to T,$$

yielding a nontrivial subtype

$$\bigcap_{T:\mathbb{U}_1} T \to T \supsetneq \bigcap_{T:\mathbb{U}_\ell} T \to T \ni \lambda x.x.$$

[since all types respect $\succcurlyeq$, Top is the only one that can contain $\circlearrowright$.]

p. 12

this interlude is just here as an overall reminder that union types can be pretty nonintuitive.

if memory serves, Aleksey and Alexei presented a similar example of the same phenomenon a few years back.

incidentally, although $\tilde{U}_1$ does have a lot of messy overlaps, it's not a singleton. (e.g., $0 \neq 1 \in \tilde{U}_1$. at present, this variant of type theory has no "bare" quotients.)

p. 13

we now see a real, non-pseudocode definition of f_1.

a subtle point here is that $\{t\}_T \sim \circlearrowright$ for every closed term $t \notin T$. (type operators are actually non-canonical in this variant of type theory.) consequently, either $x \in \tilde{U}_1$ and $xm(\{x\}_{\tilde{U}_1}) \Downarrow \mathsf{inl}(e)$ for some e, or $x \notin \tilde{U}_1$ and hence $xm(\{x\}_{\tilde{U}_1}) \sim xm(\circlearrowright) \sim \circlearrowright$. the inr branch of the case can thus never be triggered.

[to see why $xm(\circlearrowright) \sim \circlearrowright$, consider what would happen if $xm(\circlearrowright) \Downarrow o(\cdots)$ for some canonical operator o. since $\mathbb{N} \succcurlyeq \circlearrowright$, $xm(\mathbb{N}) \Downarrow o(\cdots)$ (possibly with different arguments), whence o must be inl. however, $xm(\mathsf{Void}) \Downarrow o(\cdots)$, too, whence o must be inr, contradiction.]

p. 13 (cont'd)

f_1 happens to be a bellwether of $\bigcap_{T:\mathbb{U}_1} T \to T$, which is of some independent interest since we did **not** have to assume that the constituent xm was commensurately special.

for example, everything still works if xm is actually an inhabitant of a higher level of excluded middle, like $\prod_{T:\mathbb{U}_9} T + \neg T$.

aside... the $xm(\{x\}_T)$ trick can be extended to obtain, within the object theory, constructions of a surprisingly wide variety of bellwethers.

are all of the bellwethers that are needed for Howe's framework similarly intrinsic, in some sense? that question hasn't been investigated in full detail yet, but it's currently looking like that probably is the case.

p. 14

rank is a function in the metatheory, not the object theory.

this page demonstrates the existence of a $t_\circ$ and $T_\circ$ meeting the conditions on p. 25. (explicitly, $t_\circ = \mathbb{U}_1$, $T_\circ = \{\mathbb{U}_1\}_{\mathbb{U}_2}$, and $\ell = 3$.)

thus, $\bigcap_{T:\mathbb{U}_1} T \to T \supsetneq \bigcap_{T:\mathbb{U}_3} T \to T$.

p. 14 (cont'd)

with a related argument, we can avoid skipping a universe level and, generalizing that to higher universes, go on to get an infinite descending chain

$$\bigcap_{T:\mathbb{U}_1} T \to T \supsetneq \bigcap_{T:\mathbb{U}_2} T \to T \supsetneq \bigcap_{T:\mathbb{U}_3} T \to T \supsetneq \cdots$$

this contrasts sharply with the

$$\bigcap_{T:\mathbb{U}_1} T \to T \equiv \bigcap_{T:\mathbb{U}_2} T \to T \equiv \bigcap_{T:\mathbb{U}_3} T \to T \equiv \cdots$$

that we saw in the orange pages (the Nuprl 5 type theory).

pp. 15–16

although not strictly necessary, i find this to be a more satisfying proof that $\bigcap_{T:\mathbb{U}_1} T \to T$ has nontrivial subtypes.

the idea is to find disjoint, nonempty subtypes S_0 and S_1.

[if $\mathsf{Void} \subsetneq S_0 \subseteq T$, $\mathsf{Void} \subsetneq S_1 \subseteq T$, and $S_0 \pitchfork^{\cdot} S_1$, then clearly $\mathsf{Void} \subsetneq S_0 \subsetneq T$ (and likewise for S_1).]

$$S_i \triangleq \Big(\bigcap_{T:\mathbb{U}_1} T \to T\Big) \cap \big(\{\mathbb{U}_1\}^\flat \to \{i\}^\flat\big)$$

it would have segued nicely into a discussion of separativeness, which can be used to generalize such constructions.

pp. 15–16 (cont'd)

$T \Cap^{\cdot} T'$ means that T and T' are disjoint, i.e., that $T \cap T' \equiv \mathsf{Void}$. likewise, $T \cap^{\cdot} T'$ means that $T \cap T' \not\equiv \mathsf{Void}$. the red dot just emphasizes that it's being read as a proposition.

if we define $D \triangleq \bigcap_{T:\mathbb{U}_1} T \to T$, then

$$\begin{aligned} S_0 \cap S_1 &\equiv (D \cap (\{\mathbb{U}_1\}^\flat \to \{0\}^\flat)) \cap (D \cap (\{\mathbb{U}_1\}^\flat \to \{1\}^\flat)) \\ &\equiv D \cap (\{\mathbb{U}_1\}^\flat \to \{0\}^\flat) \cap (\{\mathbb{U}_1\}^\flat \to \{1\}^\flat). \end{aligned}$$

the singletons $\{\mathbb{U}_1\}^\flat \to \{0\}^\flat$ and $\{\mathbb{U}_1\}^\flat \to \{1\}^\flat$ have no overlap, since, in general, $A \cap^{\cdot} A'$ and $B \Cap^{\cdot} B'$ imply $A \to B \Cap^{\cdot} A' \to B'$.

consequently, $S_0 \cap S_1 \equiv D \cap \mathsf{Void} \equiv \mathsf{Void}$.

pp. 15–16 (cont'd)

[to see why $A \mathrel{\dot{\cap}} A' \wedge B \mathrel{\dot{⩀}} B'$ implies $A \to B \mathrel{\dot{⩀}} A' \to B'$, suppose that we had an $a \in A \cap A'$. any member f of $(A \to B) \cap (A' \to B')$ would then have to satisfy both $f\,a \in B$ and $f\,a \in B'$, which is impossible by disjointness.]

[interestingly, in this type theory, we also get

$$A \mathrel{\dot{⩀}} A' \;\Rightarrow\; A \to B \mathrel{\dot{\cap}} A' \to B'.$$

the λ term on p. 16 gives a hint of how an inhabitant of the rhs can be constructed.]

pp. 15–16 (cont'd)

$\{\cdot\}^\flat$ is meta-notation.

when it pertains, $\{t\}^\flat$ has the nice property that all of its members are $\sim$. consequently, $\{t\}^\flat$ is a minimal singleton (no nontrivial subtypes).

the λ term on p. 16 demonstrates that S_0 is nonempty.

the corresponding inhabitant of S_1 would use a 1 instead of a 0 in the inr branch.

name disambiguation...

Douglas J. Howe
Mark Bickford
Aleksey Nogin
Alexei Kopylov

the semantics of the second type theory is chiefly due to Howe. this particular way of presenting it—using infinite trees of singletons, bellwethers, separativeness, etc.—is just my own take on what's fundamentally going on in his framework. [it developed while wrestling with order-theoretic topics that pop up when you try to reconcile $\bigcap$ and $\bigcup$ with that structure. (a foray into the question of exactly which recursive types are consistent here was also a big influence.)]

the exact boundaries of "NoNamePrl" were intentionally not pinned down in the seminar.

things are still evolving, but, at present, there are actually two sub-variants, one that has "boxed" quotients but not $\bigcap$ and $\bigcup$, and another that has $\bigcap$ and $\bigcup$ but no form of quotienting (yet). this material obviously requires the latter.

there isn't currently a formal system of rules in place that would allow the arguments of the second part of the talk to be done within its object theory, but it seems plausible that that could be done eventually. it would likely take a fair amount of work and care.

in a greater sense, "NoNamePrl" would be this future system (hopefully with a better name), one fully embracing the principles on p. 10.

strictly speaking, though, the existing proof takes place within the metatheory (the semantics)—just rendered here using type-theoretic notation instead of α, γ, etc.