Synthetic Topology in NuPRL

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What is Synthetic Topology?

Branch of topology, designed to export topological results into other fields.

Escardó, Martín. 2004. *Synthetic Topology of Data Types and Classical Spaces*. ENTCS 87.

What is Synthetic Topology?

“1. to explain what has been done in classical topology in conceptual terms,

2. to provide one-line, enlightening proofs of the theorems that constitute the core of the theory, and

3. to smoothly export topological concepts and theorems to unintended situations, keeping the synthetic proofs unmodified.”

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Classical Topology

(X, T_X)

X - set of points,

T_X - family of *open sets* of X

closed under:

finite intersection

arbitrary union

Classical Topology

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Classical Topology

Continuous functions, $f : (X, T_X) \to (Y, T_Y)$

Maps points forward

$$f \quad : X \to Y$$

Maps open sets backwards

$$f^{-1} : T_Y \to T_X$$

Sierpinski Space

In what follows, **Σ** plays an important role.

$$\mathbf{\Sigma} = \{ \top, \bot \}$$

$$T_{\Sigma} = \{ \{\}, \{ \top \}, \{ \top, \bot \} \}$$

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We'll think of **Σ** as **semidecidable truth values**. We'll think of functions into **Σ** as **semidecidable sets.**

Sierpinski Space

There is a correspondence between

open sets of X

and

continuous functions from X to **Σ**.

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So topology is really just about continuous functions into **Σ**. Topology is about **semidecidable sets**.

Synthetic Topology

Ingredients for (synthetic) topology:

1. Spaces
2. Functions
3. Sierpinski Space

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Types are spaces, and each type has a built in "topology", whose open sets correspond to the functions from that type into **Σ**.

Sierpinski Type

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Σ must represent semidecidable truth values.

Sierpinski Type in NuPRL

$$\boldsymbol{\Sigma} = (\mathbf{N} \rightarrow \mathbf{B}) \,//\, \sim$$

$$(f \sim g) \text{ iff } (\forall n.\, f\, n = \mathbf{ff}) \Leftrightarrow (\forall n.\, g\, n = \mathbf{ff})$$

Sierpinski Type in NuPRL

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$$\bot = \lambda n.\ \mathbf{ff}$$

$$\top = \lambda n.\ \mathbf{tt}$$

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Semidecidable.

Sierpinski Type

$$(\vee) : \boldsymbol{\Sigma} \to \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}$$

$$(\wedge) : \boldsymbol{\Sigma} \to \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}$$

Sierpinski Type

$(\vee) : \mathbf{\Sigma} \rightarrow \mathbf{\Sigma} \rightarrow \mathbf{\Sigma}$

$f \vee g = \lambda x.\ (f\ x\ ||\ g\ x)$

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$(\vee) : \mathbf{\Sigma} \to \mathbf{\Sigma} \to \mathbf{\Sigma}$

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f ⋀ g = λx. (or(i<x. f i) && or(i<x. g i))

Sierpinski Type

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f ∨ g = λx. (f x || g x)

$(\wedge) : \mathbf{\Sigma} \to \mathbf{\Sigma} \to \mathbf{\Sigma}$

f ∧ g = λx. (or(i<x. f i) && or(i<x. g i))

(can’t just do pointwise &&)

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Countable join.

cjoin : $(\mathbf{N} \to \mathbf{\Sigma}) \to \mathbf{\Sigma}$

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dovetail : $(\mathbf{N} \to \mathbf{N} \to \mathbf{B}) \to (\mathbf{N} \to \mathbf{B})$

dovetail f n = or(i < n. or(j < n. f i j))

Sierpinski Type

Countable join.

cjoin : ($\mathbf{N} \to \mathbf{\Sigma}$) $\to \mathbf{\Sigma}$

cjoin = dovetail

Requires LEMMA:

$$\frac{X \subseteq \text{Base}}{X \to (Y//E) \subseteq (X \to Y)//E'}$$

dovetail : ($\mathbf{N} \to \mathbf{N} \to \mathbf{B}$) $\to$ ($\mathbf{N} \to \mathbf{B}$)

dovetail f n = or(i < n. or(j < n. f i j))

Synthetic Topology

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Let Open(X) be the type $X \to \mathbf{\Sigma}$.

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Synthetic Topology

An **open** of X is a function from X to Σ.

Let Open(X) be the type $X \to \Sigma$.

Given $f : X \to Y$

$A : \mathrm{Open}(Y)$

All functions are continuous.

We get $A \circ f : \mathrm{Open}(X)$ by composition.

Synthetic Topology

Intersection of opens == pointwise meet.

Union of opens == pointwise join.

Therefore:

Opens have finite intersections

and countable unions.

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Such that $(\exists X.A = \bot)$ iff $(\forall x.\ A\ x = \bot)$

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Such that $(\exists X.A = \bot)$ iff $(\forall x.\ A\ x = \bot)$

Countable unions iff **N** is overt.

Compact Spaces

We also have **compact intersections**.

A space X is **compact** iff there is a function:

$$\forall X : Open(X) \text{ -> } \mathbf{\Sigma}$$

Such that $(\forall X.A = \top)$ iff $(\forall x.\ A\ x = \top)$

All finite sets are compact (and overt).

Compactness is dual to overtness.

Changing Assumptions

$$\mathbf{\Sigma} = (\mathbf{N} \to \mathbf{B}) \mathbin{//} \sim$$

implies countable union,

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Yes, if $X \subseteq \text{Base}$ and $X \twoheadrightarrow X \times X$.

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Can we weaken this?

Subspaces

If $X \subseteq Y$ then X has a topology that is at least as fine as the subspace topology of Y, because $\mathrm{Open}(Y) \subseteq \mathrm{Open}(X)$.

X is an **open subspace** of Y iff there is a

$A : \mathrm{Open}(Y)$ s.t. $X = \{ y : Y \mid A\ x = \top \}$.

Open Subspace vs Overt Subspace

If X is an open subspace of Y,
and Y is overt,
then X is overt.

If X is an overt subspace of Y,
and Y has semidecidable equality,
then X is an open subspace of Y.

Open Subspace vs Overt Subspace

If X is an open subspace of Y,
and Y is overt,
then X is overt.

Not yet verified.
But simple proofs.

If X is an overt subspace of Y,
and Y has semidecidable equality,
then X is an open subspace of Y.

Limitations

We don’t have any **continuity principle**.

It should be the case that functions

$$(\mathbf{N} \to \mathbf{B}) \to \mathbf{B}$$

can depend only on finitely many values of the input. But that doesn’t come for free.

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We probably can’t show $(\mathbf{N} \to \mathbf{B})$ is compact.

We can’t show opens on $\mathbf{R}$ are “open sets”.

Limitations

We don't have **Markov's principle**.

We can't assume that just because

$A\ x \sim \top$

then there must exist some n : **N** such that

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then there must exist some n : **N** such that

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This is probably incompatible with (weak) continuity principles.

Conclusion

We've taken the first steps to formalizing synthetic topology in NuPRL.

There is much work yet to do.

Especially:

- Looking at Tychonoff's theorem.
- Looking for something similar for overtness.