```
split_factor2_char  THM

⊢  ∀k:{2...}, g:({2..k⁻}→ℕ), x,y:{2..k⁻}.
|  x·y<k
|  ⇒
|  x<y
|  ⇒
|  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(split_factor2(g; x; y))
|  & split_factor2(g; x; y)(x·y) = 0
|  & (∀u:{2..k⁻}. x·y<u ⇒ split_factor2(g; x; y)(u) = g(u))
| by {After letting h = split_factor2(g; x; y) and elaborating the
|     four cases from its definition, the only work is showing that it
|     factors the same number as g
|     }
|    SideProof 1
|
1. k : {2...}
2. g : {2..k⁻}→ℕ
3. x : {2..k⁻}
4. y : {2..k⁻}
5. x·y<k
6. x<y
7. x<x·y
8. y<x·y
9. h : {2..k⁻}→ℕ
10. h = split_factor2(g; x; y)
11. h(x·y) = 0 ∈ ℕ
12. h(x) = g(x)+g(x·y) ∈ ℕ
13. h(y) = g(y)+g(x·y) ∈ ℕ
14. ∀u:{2..k⁻}. ¬u = x ⇒ ¬u = y ⇒ ¬u = x·y ⇒ h(u) = g(u)
⊢  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(h)
| by Split the product at x, y, and x·y using
|    Thm*  ∀a,c,b:ℤ, f:({a..b⁻}→ℤ).
|          a≤c
|          ⇒
|          c<b ⇒ Π↑{a..b⁻}(f) = Π↑{a..c⁻}(f)·c↑f(c)·Π↑{c+1..b⁻}(f)
|    {eval_factorization_split_pluck}
|
⊢  Π↑{2..x⁻}(g)·x↑g(x)·Π↑{x+1..y⁻}(g)·y↑g(y)·Π↑{y+1..x·y⁻}(g)
|                                            ·(x·y)↑g(x·y)
|                                             ·Π↑{x·y+1..k⁻}(g)
|  =
|  Π↑{2..x⁻}(h)·x↑h(x)·Π↑{x+1..y⁻}(h)·y↑h(y)·Π↑{y+1..x·y⁻}(h)
|                                            ·(x·y)↑h(x·y)
|                                             ·Π↑{x·y+1..k⁻}(h)
| by {Factor out the sections falling between x, y, and x·y, which
|     agree by Hyp:14
|     }
|    SideProof 2
|
⊢  x↑g(x)·y↑g(y)·(x·y)↑g(x·y) = x↑h(x)·y↑h(y)·(x·y)↑h(x·y)
| by {Eliminate h in favor of g. }
|    Subst: Hyp:11 ...wℕ THEN Subst: Hyp:12 ...wℕ
|    THEN
|    Subst: Hyp:13 ...wℕ
```

```
|
⊢  x↑g(x)·y↑g(y)·(x·y)↑g(x·y)
|  =
|  x↑(g(x)+g(x·y))·y↑(g(y)+g(x·y))·(x·y)↑0
| by {normalize factors into powers of x and y }
|    Rewrite by
|    Thm*  ∀a,b,c:ℕ. (a·b)↑c = a↑c·b↑c    {exp_reduce1} ...wℕ
|    THEN
|    Rewrite by
|    Thm*  ∀i:ℤ, x,y:ℕ. i↑x·i↑y = i↑(x+y)    {sum_exponent} ...wℕ
|
⊢  x↑g(x)·y↑g(y)·x↑g(x·y)·y↑g(x·y)
   =
   x↑g(x)·x↑g(x·y)·y↑g(y)·y↑g(x·y)·x↑0·y↑0
  by Rewrite by Thm*  ∀x:ℤ. x↑0 = 1    {exponent_zero} ...ℕ
END
---

split_factor2_char Side Proof 1  THM

⊢  ∀k:{2...}, g:({2..k⁻}→ℕ), x,y:{2..k⁻}.
|  x·y<k
|  ⇒
|  x<y
|  ⇒
|  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(split_factor2(g; x; y))
|  & split_factor2(g; x; y)(x·y) = 0
|  & (∀u:{2..k⁻}. x·y<u ⇒ split_factor2(g; x; y)(u) = g(u))
| by UnivCD ...wℕ
|    THEN
|    Inst: Thm*  ∀i,j:{2...}. i<i·j & j<i·j    {factors_smaller}
|    Using:[x | y]
|
1. k : {2...}
2. g : {2..k⁻}→ℕ
3. x : {2..k⁻}
4. y : {2..k⁻}
5. x·y<k
6. x<y
7. x<x·y & y<x·y
⊢  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(split_factor2(g; x; y))
|  & split_factor2(g; x; y)(x·y) = 0
|  & (∀u:{2..k⁻}. x·y<u ⇒ split_factor2(g; x; y)(u) = g(u))
| by Analyze-1 THEN Analyze
|
|\
| 7. x<x·y
| 8. y<x·y
| ⊢  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(split_factor2(g; x; y))
| | by Let (h = split_factor2(g; x; y)) ...wℕ
| |    THEN
| |    ((h(x·y) = 0 ∈ ℕ
| |      & h(x) = g(x)+g(x·y) ∈ ℕ
| |      & h(y) = g(y)+g(x·y) ∈ ℕ
| |      & (∀u:{2..k⁻}.
```

```
| |         ¬u = x ∈ ℤ
| |         ⇒
| |         ¬u = y ∈ ℤ ⇒ ¬u = x·y ∈ ℤ ⇒ h(u) = g(u) ∈ ℕ)
| |        By (Rewrite by
| |            h = split_factor2(g; x; y) ∈ {2..k⁻}→ℕ ...wℕ
| |            THEN
| |            Def of split_factor2(g; x; y)
| |            THEN
| |            Reduce Concl ...ℕ THEN SplitOnConclITEs ...ℕ))
| |     THEN
| |     Autoℕ)
| |
| 9. h : {2..k⁻}→ℕ
| 10. h = split_factor2(g; x; y)
| 11. h(x·y) = 0 ∈ ℕ
| 12. h(x) = g(x)+g(x·y) ∈ ℕ
| 13. h(y) = g(y)+g(x·y) ∈ ℕ
| 14. ∀u:{2..k⁻}. ¬u = x ⇒ ¬u = y ⇒ ¬u = x·y ⇒ h(u) = g(u)
| ⊢  Π↑{2..k⁻}(g) = Π↑{2..k⁻}(h)  by <Unproved PREMISE>
|
.\
. 7. x<x·y
. 8. y<x·y
. ⊢  split_factor2(g; x; y)(x·y) = 0
.    & (∀u:{2..k⁻}. x·y<u ⇒ split_factor2(g; x; y)(u) = g(u))
.   by Autoℕ
.      THEN
.      Def of split_factor2(<term>; <term>; <term>) THEN Reduce Concl
.      THEN
.      SplitOnConclITEs ...ℕ
END
---

split_factor2_char Side Proof 2  THM

1. k : {2...}
2. g : {2..k⁻}→ℕ
3. x : {2..k⁻}
4. y : {2..k⁻}
5. x·y<k
6. x<y
7. x<x·y
8. y<x·y
9. h : {2..k⁻}→ℕ
10. h = split_factor2(g; x; y)
11. h(x·y) = 0 ∈ ℕ
12. h(x) = g(x)+g(x·y) ∈ ℕ
13. h(y) = g(y)+g(x·y) ∈ ℕ
14. ∀u:{2..k⁻}. ¬u = x ⇒ ¬u = y ⇒ ¬u = x·y ⇒ h(u) = g(u)
⊢  Π↑{2..x⁻}(g)·x↑g(x)·Π↑{x+1..y⁻}(g)·y↑g(y)·Π↑{y+1..x·y⁻}(g)
|                                                 ·(x·y)↑g(x·y)
|                                                  ·Π↑{x·y+1..k⁻}(g)
|  =
|  Π↑{2..x⁻}(h)·x↑h(x)·Π↑{x+1..y⁻}(h)·y↑h(y)·Π↑{y+1..x·y⁻}(h)
|                                                 ·(x·y)↑h(x·y)
```

```
|                                                     ·Π↑{x·y+1..k⁻}(h)
| by MulFactorOut
|     [Π↑{2..x⁻}(g),Π↑{2..x⁻}(h)
|     ;Π↑{x+1..y⁻}(g),Π↑{x+1..y⁻}(h)
|     ;Π↑{y+1..x·y⁻}(g),Π↑{y+1..x·y⁻}(h)
|     ;Π↑{x·y+1..k⁻}(g),Π↑{x·y+1..k⁻}(h)]
|     Concl ...aℕ
|
|\
| .....factor equality .....
| ⊢  Π↑{2..x⁻}(g) = Π↑{2..x⁻}(h)
|   by Analyze ...ℕ THEN New:u FunExtensionality
|      THEN
|      BackThru: Hyp:14 ...ℕ
|
|\
| .....factor equality .....
| ⊢  Π↑{x+1..y⁻}(g) = Π↑{x+1..y⁻}(h)
|   by Analyze ...ℕ THEN New:u FunExtensionality
|      THEN
|      BackThru: Hyp:14 ...ℕ
|
|\
| .....factor equality .....
| ⊢  Π↑{y+1..x·y⁻}(g) = Π↑{y+1..x·y⁻}(h)
|   by Analyze ...ℕ THEN New:u FunExtensionality
|      THEN
|      BackThru: Hyp:14 ...ℕ
|
|\
| .....factor equality .....
| ⊢  Π↑{x·y+1..k⁻}(g) = Π↑{x·y+1..k⁻}(h)
|   by Analyze ...ℕ THEN New:u FunExtensionality
|      THEN
|      BackThru: Hyp:14 ...ℕ
|
.\
. ⊢  x↑g(x)·y↑g(y)·(x·y)↑g(x·y) = x↑h(x)·y↑h(y)·(x·y)↑h(x·y)
.   by <Unproved PREMISE>
END
```