hanoi_general_exists_lemma2 THM

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⊢  ∀n:ℕ⁺, a:ℤ, z:{a...}, m:{a...z-1}, f,g:({1...n}→Peg).
|  f(n) ≠ g(n)
|  ⇒
|  (∃s1:({a...m}→{1...n-1}→Peg), s2:({m+1...z}→{1...n-1}→Peg).
|   s1 is a Hanoi(n-1 disk) seq on a..m
|   & s1(a) = f ∈ {1...n-1}→Peg
|   & s2 is a Hanoi(n-1 disk) seq on m+1..z
|   & s2(z) = g ∈ {1...n-1}→Peg
|   & s1(m) = s2(m+1)
|   & (∀i:{1...n-1}. s1(m,i) ≠ f(n) & s2(m+1,i) ≠ g(n)))
|  ⇒
|  (∃r1:({a...m}→{1...n}→Peg), r2:({m+1...z}→{1...n}→Peg).
|   (r1 @(m) r2) is a Hanoi(n disk) seq on a..z
|   & r1(a) = f
|   & r2(z) = g)  by Auto THEN New:s1 Analyze-1 THEN New:s2 Analyze-1
|
1. n : ℕ⁺
2. a : ℤ
3. z : {a...}
4. m : {a...z-1}
5. f : {1...n}→Peg
6. g : {1...n}→Peg
7. f(n) ≠ g(n)
8. s1 : {a...m}→{1...n-1}→Peg
9. s2 : {m+1...z}→{1...n-1}→Peg
10. s1 is a Hanoi(n-1 disk) seq on a..m
11. s1(a) = f ∈ {1...n-1}→Peg
12. s2 is a Hanoi(n-1 disk) seq on m+1..z
13. s2(z) = g ∈ {1...n-1}→Peg
14. s1(m) = s2(m+1)
15. ∀i:{1...n-1}. s1(m,i) ≠ f(n) & s2(m+1,i) ≠ g(n)
⊢  ∃r1:({a...m}→{1...n}→Peg), r2:({m+1...z}→{1...n}→Peg).
|  (r1 @(m) r2) is a Hanoi(n disk) seq on a..z & r1(a) = f & r2(z) = g
| by Witness: s1(?) {to n-1} ⊕ f {to n} | s2(?) {to n-1} ⊕ g {to n}
|
|\
| ⊢  ((s1(?) {to n-1} ⊕ f {to n}) @(m) (s2(?) {to n-1} ⊕ g {to n}))
| |  is a Hanoi(n disk) seq on a..z
| | by BackThru:
| |    Thm*  ∀n:ℕ, a,z:ℤ, m:{a...z-1}, s1:({a...m}→{1...n}→Peg),
| |           s2:({m+1...z}→{1...n}→Peg).
| |          (∃k:{1...n}. Moving disk k of n takes s1(m) to s2(m+1))
| |          ⇒
| |          s1 is a Hanoi(n disk) seq on a..m
| |          ⇒
| |          s2 is a Hanoi(n disk) seq on m+1..z
| |          ⇒
| |          (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
| |    {hanoi_seq_join_seq}
| |
| |\
| | ⊢  ∃k:{1...n}.
| | |  Moving disk k of n
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| | |  takes (s1(?) {to n-1} ⊕ f {to n})(m)
| | |  to (s2(?) {to n-1} ⊕ g {to n})(m+1)
| | | by Witness: n THEN 2 Times Analyze
| | |
| | |\
| | | 16. i : {1...n}
| | | ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i)
| | | |  =
| | | |  (s2(?) {to n-1} ⊕ g {to n})(m+1,i)
| | | |  ⟺
| | | |  i ≠ n  by Decide: i = n ∈ ℤ
| | | |
| | | |\
| | | | 17. i = n
| | | | ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i)
| | | | |  =
| | | | |  (s2(?) {to n-1} ⊕ g {to n})(m+1,i)
| | | | |  ⟺
| | | | |  i ≠ n
| | | | | by m ∈ {a...m}  Asserted
| | | | |    THEN
| | | | |    Rewrite by Thm: hanoi_seq_deepen_higheq
| | | | |
| | | | 18. m ∈ {a...m}
| | | | ⊢  f(i) = g(i) ⟺ i ≠ n  by Rewrite by i = n ∈ {1...n}
| | | | |
| | | | ⊢  f(n) = g(n) ⟺ n ≠ n  by Auto
| | | | |
| | | | 19. f(n) = g(n)
| | | | ⊢  n ≠ n  by SimilarTo: Hyp:7
| | | |
| | | .\
| | | . 17. ¬i = n
| | | . ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i)
| | | . |  =
| | | . |  (s2(?) {to n-1} ⊕ g {to n})(m+1,i)
| | | . |  ⟺
| | | . |  i ≠ n
| | | . | by Guarding (<fun>(m))
| | | . |    Rewrite by Thm: hanoi_seq_deepen_loweq
| | | . |
| | | . |\
| | | . | ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i) ∈ Peg
| | | . |   by With ({1...n-1}→Peg) Analyze
| | | . |
| | | . .\
| | | . . 18. i ≠ n

| | | . . ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i) = s2(m+1,i)
| | | . . | by m ∈ {a...m}  Asserted
| | | . . |    THEN
| | | . . |    Rewrite by Thm: hanoi_seq_deepen_loweq
| | | . . |
| | | . . 19. m ∈ {a...m}
| | | . . ⊢  s1(m,i) = s2(m+1,i)  by Rewrite by Hyp:14
| | |
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| | .\
| | . 16. i : {1...n-1}
| | . ⊢  (s1(?) {to n-1} ⊕ f {to n})(m,i)
| | . |  ≠
| | . |  (s1(?) {to n-1} ⊕ f {to n})(m,n)
| | . |  & (s2(?) {to n-1} ⊕ g {to n})(m+1,i)
| | . |    ≠
| | . |    (s2(?) {to n-1} ⊕ g {to n})(m+1,n)
| | . | by m ∈ {a...m}  Asserted
| | . |    THEN
| | . |    Guarding (<fun>(n)) Rewrite by Thm: hanoi_seq_deepen_loweq
| | . |    THEN
| | . |    Rewrite by Thm: hanoi_seq_deepen_higheq
| | . |
| | . |\
| | . | 17. m ∈ {a...m}
| | . | ⊢  s1(m,i) ≠ f(n)  by BackThru: Hyp:15
| | . |
| | . .\
| | . . 17. m ∈ {a...m}
| | . . ⊢  s2(m+1,i) ≠ g(n)  by BackThru: Hyp:15
| |
| |\
| | ⊢  (s1(?) {to n-1} ⊕ f {to n}) is a Hanoi(n disk) seq on a..m
| |   by BackThru: Thm: hanoi_seq_deepen_seq
| |
| .\
| . ⊢  (s2(?) {to n-1} ⊕ g {to n}) is a Hanoi(n disk) seq on m+1..z
| .   by BackThru:
| .      Thm*  ∀a,z:ℤ, n:ℕ, s:({a...z}→{1...n}→Peg), n':ℕ.
| .            n≤n'
| .            ⇒
| .            (∀h:({n+1...n'}→Peg).
| .             s is a Hanoi(n disk) seq on a..z
| .             ⇒
| .             (s(?) {to n} ⊕ h {to n'})
| .             is a Hanoi(n' disk) seq on a..z)
| .      {hanoi_seq_deepen_seq}
|
|\
| ⊢  (s1(?) {to n-1} ⊕ f {to n})(a) = f
| | by FunExtensionality THEN Decide: x<n
| |
| |\
| | 16. x : {1...n}
| | 17. x<n
| | ⊢  (s1(?) {to n-1} ⊕ f {to n})(a,x) = f(x)
| | | by Rewrite by Thm: hanoi_seq_deepen_loweq
| | |
| | ⊢  s1(a,x) = f(x)  by Analyze
| |
| .\
| . 16. x : {1...n}
| . 17. ¬x<n
| . ⊢  (s1(?) {to n-1} ⊕ f {to n})(a,x) = f(x)
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| .   by Rewrite by Thm: hanoi_seq_deepen_higheq
|
.\
. ⊢  (s2(?) {to n-1} ⊕ g {to n})(z) = g
. | by FunExtensionality THEN Decide: x<n
. |
. |\
. | 16. x : {1...n}
. | 17. x<n
. | ⊢  (s2(?) {to n-1} ⊕ g {to n})(z,x) = g(x)
. | | by Rewrite by Thm: hanoi_seq_deepen_loweq
. | |
. | ⊢  s2(z,x) = g(x)  by Analyze
. |
. .\
. . 16. x : {1...n}
. . 17. ¬x<n
. . ⊢  (s2(?) {to n-1} ⊕ g {to n})(z,x) = g(x)
. .   by Rewrite by Thm: hanoi_seq_deepen_higheq
END
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