```
hanoi_sol2_ala_general  THM

⊢  ∀n:ℕ, p,q:Peg.
|  p ≠ q
|  ⇒
|  (∀a:ℤ.
|   ∃z:{a...}, s:({a...z}→{1...n}→Peg).
|   s is a Hanoi(n disk) seq on a..z & s(a) = (λi.p) & s(z) = (λi.q))
| by NatInd Concl
|
|\
| 1. p : Peg
| 2. q : Peg
| 3. p ≠ q
| 4. a : ℤ
| ⊢  ∃z:{a...}, s:({a...z}→{1...0}→Peg).
| |  s is a Hanoi(0 disk) seq on a..z & s(a) = (λi.p) & s(z) = (λi.q)
| | by Witness: a | λx,i. whatever THEN Reduce Concl
| |
| |\
| | ⊢  (λx,i. whatever) is a Hanoi(0 disk) seq on a..a  by Analyze
| |
| |\
| | ⊢  (λi.whatever) = (λi.p)  by FunExtensionality
| |
| .\
| . ⊢  (λi.whatever) = (λi.q)  by FunExtensionality
|
.\
. 1. n : ℤ
. 2. 0<n
. 3. ∀p,q:Peg.
. |  p ≠ q
. |  ⇒
. |  (∀a:ℤ.
. |   ∃z:{a...}, s:({a...z}→{1...n-1}→Peg).
. |   s is a Hanoi(n-1 disk) seq on a..z
. |   & s(a) = (λi.p)
. |   & s(z) = (λi.q))
. 4. p : Peg
. 5. q : Peg
. 6. p ≠ q
. 7. a : ℤ
. ⊢  ∃z:{a...}, s:({a...z}→{1...n}→Peg).
. |  s is a Hanoi(n disk) seq on a..z & s(a) = (λi.p) & s(z) = (λi.q)
. | by ∃z:{a...}, m:{a...z-1}, s1:({a...m}→{1...n}→Peg),
. |     s2:({m+1...z}→{1...n}→Peg).
. |    (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
. |    & s1(a) = (λi.p)
. |    & s2(z) = (λi.q)
. |    Asserted
. |
. |\
. | .....assertion .....
. | ⊢  ∃z:{a...}, m:{a...z-1}, s1:({a...m}→{1...n}→Peg),
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. | |   s2:({m+1...z}→{1...n}→Peg).
. | |  (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
. | |  & s1(a) = (λi.p)
. | |  & s2(z) = (λi.q)
. | | by ¬p = otherPeg(p; q) | ¬otherPeg(p; q) = q  Asserted
. | |    THEN
. | |    (Inst: 3 Using:[p | otherPeg(p; q) | a]
. | |     THEN
. | |     Inst: 3 Using:[otherPeg(p; q) | q | m+1]
. | |     THEN
. | |     Witness: z | m)
. | |
. | |\
. | | .....assertion .....
. | | ⊢  ¬p = otherPeg(p; q)
. | |   by BackThru:
. | |      Thm*  ∀x,y:Peg. x ≠ y ⇒ x ≠ otherPeg(x; y)
. | |      {hanoi_otherpeg_diff3}
. | |
. | |\
. | | .....assertion .....
. | | 8. ¬p = otherPeg(p; q)
. | | ⊢  ¬otherPeg(p; q) = q
. | |   by BackThru:
. | |      Thm*  ∀x,y:Peg. x ≠ y ⇒ otherPeg(x; y) ≠ y
. | |      {hanoi_otherpeg_diff2}
. | |
. | .\
. | . 8. ¬p = otherPeg(p; q)
. | . 9. ¬otherPeg(p; q) = q
. | . 10. m : {a...}
. | . 11. s1 : {a...m}→{1...n-1}→Peg
. | . 12. s1 is a Hanoi(n-1 disk) seq on a..m
. | . 13. s1(a) = (λi.p)
. | . 14. s1(m) = (λi.otherPeg(p; q))
. | . 15. z : {m+1...}
. | . 16. s2 : {m+1...z}→{1...n-1}→Peg
. | . 17. s2 is a Hanoi(n-1 disk) seq on m+1..z
. | . 18. s2(m+1) = (λi.otherPeg(p; q))
. | . 19. s2(z) = (λi.q)
. | . ⊢  ∃s1:({a...m}→{1...n}→Peg), s2:({m+1...z}→{1...n}→Peg).
. | . |  (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
. | . |  & s1(a) = (λi.p)
. | . |  & s2(z) = (λi.q)
. | . | by BackThru:
. | . |    Thm*  ∀n:ℕ⁺, a:ℤ, z:{a...}, m:{a...z-1},
. | . |           f,g:({1...n}→Peg).

. | . |          f(n) ≠ g(n)
. | . |          ⇒
. | . |          (∃s1:({a...m}→{1...n-1}→Peg),
. | . |            s2:({m+1...z}→{1...n-1}→Peg).
. | . |           s1 is a Hanoi(n-1 disk) seq on a..m
. | . |           & s1(a) = f ∈ {1...n-1}→Peg
. | . |           & s2 is a Hanoi(n-1 disk) seq on m+1..z
. | . |           & s2(z) = g ∈ {1...n-1}→Peg
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. | . |           & s1(m) = s2(m+1)
. | . |           & (∀i:{1...n-1}. s1(m,i) ≠ f(n) & s2(m+1,i) ≠ g(n)))
. | . |          ⇒
. | . |          (∃r1:({a...m}→{1...n}→Peg),
. | . |            r2:({m+1...z}→{1...n}→Peg).
. | . |           (r1 @(m) r2) is a Hanoi(n disk) seq on a..z
. | . |           & r1(a) = f
. | . |           & r2(z) = g)
. | . |    {hanoi_general_exists_lemma2}
. | . |
. | . |\
. | . | ⊢  (λi.p)(n) ≠ (λi.q)(n)  by Reduce Concl
. | . |
. | . .\
. | . . ⊢  ∃s1:({a...m}→{1...n-1}→Peg),
. | . . |   s2:({m+1...z}→{1...n-1}→Peg).
. | . . |  s1 is a Hanoi(n-1 disk) seq on a..m
. | . . |  & s1(a) = (λi.p)
. | . . |  & s2 is a Hanoi(n-1 disk) seq on m+1..z
. | . . |  & s2(z) = (λi.q)
. | . . |  & s1(m) = s2(m+1)
. | . . |  & (∀i:{1...n-1}.
. | . . |     s1(m,i) ≠ (λi.p)(n) & s2(m+1,i) ≠ (λi.q)(n))
. | . . | by Witness: s1 | s2 THEN Reduce Concl
. | . . |
. | . . |\
. | . . | 20. i : {1...n-1}
. | . . | ⊢  s1(m,i) ≠ p  by Rewrite by Hyp:14 THEN Reduce Concl
. | . . | |
. | . . | ⊢  otherPeg(p; q) ≠ p  by SimilarTo: Hyp:8
. | . . |
. | . . .\
. | . . . 20. i : {1...n-1}
. | . . . ⊢  s2(m+1,i) ≠ q  by Rewrite by Hyp:18 THEN Reduce Concl
. |
. .\
. . 8. ∃z:{a...}, m:{a...z-1}, s1:({a...m}→{1...n}→Peg),
. . |   s2:({m+1...z}→{1...n}→Peg).
. . |  (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
. . |  & s1(a) = (λi.p)
. . |  & s2(z) = (λi.q)
. . ⊢  ∃z:{a...}, s:({a...z}→{1...n}→Peg).
. . |  s is a Hanoi(n disk) seq on a..z
. . |  & s(a) = (λi.p)
. . |  & s(z) = (λi.q)  by ExistHD Hyp:-1
. . |
. . 8. z : {a...}
. . 9. m : {a...z-1}
. . 10. s1 : {a...m}→{1...n}→Peg
. . 11. s2 : {m+1...z}→{1...n}→Peg
. . 12. (s1 @(m) s2) is a Hanoi(n disk) seq on a..z
. . 13. s1(a) = (λi.p)
. . 14. s2(z) = (λi.q)
. . ⊢  ∃z:{a...}, s:({a...z}→{1...n}→Peg).
. . |  s is a Hanoi(n disk) seq on a..z
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. . |  & s(a) = (λi.p)
. . |  & s(z) = (λi.q)  by Witness: z | s1 @(m) s2
. . |
. . |\
. . | ⊢  (s1 @(m) s2)(a) = (λi.p)
. . |   by Rewrite by
. . |      Thm*  ∀n:ℕ, m,a,z:ℤ, s1:({a...m}→{1...n}→Peg),
. . |             s2:({m+1...z}→{1...n}→Peg), x:{a...m}.
. . |            (s1 @(m) s2)(x) = s1(x)
. . |      {hanoi_seq_join_part1}
. . |
. . .\
. . . ⊢  (s1 @(m) s2)(z) = (λi.q)
. . .   by Rewrite by
. . .      Thm*  ∀n:ℕ, m,a,z:ℤ, s1:({a...m}→{1...n}→Peg),
. . .             s2:({m+1...z}→{1...n}→Peg), x:{m+1...z}.
. . .            (s1 @(m) s2)(x) = s2(x)
. . .      {hanoi_seq_join_part2}
END
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