```
hanoi_sol2_lb  THM

⊢  ∀n:ℕ, p,q:Peg.
|  p ≠ q
|  ⇒
|  (∀a:ℤ, z:{a...}, s:({a...z}→{1...n}→Peg).
|   s is a Hanoi(n disk) seq on a..z & s(a) = (λi.p) & s(z) = (λi.q)
|   ⇒
|   (2^n)≤z-a+1)  by CompNatInd Concl
|
1. n : ℕ
2. ∀n1:ℕ.
|  n1<n
|  ⇒
|  (∀p,q:Peg.
|   p ≠ q
|   ⇒
|   (∀a:ℤ, z:{a...}, s:({a...z}→{1...n1}→Peg).
|    s is a Hanoi(n1 disk) seq on a..z & s(a) = (λi.p) & s(z) = (λi.q)
|    ⇒
|    (2^n1)≤z-a+1))
3. p : Peg
4. q : Peg
5. p ≠ q
6. a : ℤ
7. z : {a...}
8. s : {a...z}→{1...n}→Peg
9. s is a Hanoi(n disk) seq on a..z
10. s(a) = (λi.p)
11. s(z) = (λi.q)
⊢  (2^n)≤z-a+1
| by Compute ⌈2^n⌉ *⇒ if n=₂0→ 1 else 2•(2^(n-1)) fi
|    THEN
|    SplitITE Concl
|
.....falsecase .....
12. n ≠ 0
⊢  2•(2^(n-1))≤z-a+1
| by Inst:
|    Thm*  ∀n:ℕ⁺, p,q:Peg.
|          p ≠ q
|          ⇒
|          (∀a:ℤ, z:{a...}, s:({a...z}→{1...n}→Peg).
|           s is a Hanoi(n disk) seq on a..z
|           & s(a) = (λi.p)
|           & s(z) = (λi.q)
|           ⇒
|           (∃x:{a...z-1}, y:{x+1...z}, p',q':Peg.
|            (∀u:{a...x}. s(u,n) = p) & (∀u:{y...z}. s(u,n) = q)
|            & s(x) = (λi.p') ∈ {1...n-1}→Peg
|            & s(y) = (λi.q') ∈ {1...n-1}→Peg
|            & p ≠ p'
|            & q ≠ q'))
|    {hanoi_sol2_analemma}
|    Using:[n | p | q | a | z | s]
```

```
|    THEN
|    ExistHD Hyp:-1
|
13. x : {a...z-1}
14. y : {x+1...z}
15. p' : Peg
16. q' : Peg
17. ∀u:{a...x}. s(u,n) = p
18. ∀u:{y...z}. s(u,n) = q
19. s(x) = (λi.p') ∈ {1...n-1}→Peg
20. s(y) = (λi.q') ∈ {1...n-1}→Peg
21. p ≠ p'
22. q ≠ q'
⊢  2·(2^(n-1))≤z-a+1  by Inst: Hyp:2 Using:[n-1 | p | p' | a | x | s]
|
|\
| ⊢  s is a Hanoi(n-1 disk) seq on a..x
| | by BackThru:
| |    Thm*  ∀n':ℕ, n:{n'...}, a:ℤ, z:{a...},
| |           s:({a...z}→{1...n}→Peg).
| |          s is a Hanoi(n disk) seq on a..z
| |          ⇒
| |          (∀x:{a...z}, i:{n'+1...n}. s(a,i) = s(x,i))
| |          ⇒
| |          s is a Hanoi(n' disk) seq on a..z
| |    {hanoi_seq_shallower}
| |
| |\
| | ⊢  s is a Hanoi(n disk) seq on a..x
| |   by BackThru:
| |      Thm*  ∀n:ℕ, a,z:ℤ, s:({a...z}→{1...n}→Peg).
| |            s is a Hanoi(n disk) seq on a..z
| |            ⇒
| |            (∀a',z':{a...z}. s is a Hanoi(n disk) seq on a'..z')
| |      {hanoi_subseq}
| |      Using:[z:= z]
| |
| .\
| . 23. x1 : {a...x}
| . 24. i : {n-1+1...n}
| . ⊢  s(a,i) = s(x1,i)
| .   by Subst: i = n ∈ {1...n} THEN Rewrite by Hyp:17
|
|\
| ⊢  s(a) = (λi.p) ∈ {1...n-1}→Peg  by Inclusion Hyp:10
|
.\
. 23. (2^(n-1))≤x-a+1
. ⊢  2·(2^(n-1))≤z-a+1
. | by Inst: Hyp:2 Using:[n-1 | q' | q | y | z | s]
. |
. |\
. | .....antecedent .....
. | ⊢  q' ≠ q  by SimilarTo: Hyp:22
. |
```

```
. |\
. | ⊢  s is a Hanoi(n-1 disk) seq on y..z
. | | by BackThru:
. | |    Thm*  ∀n':ℕ, n:{n'...}, a:ℤ, z:{a...},
. | |           s:({a...z}→{1...n}→Peg).
. | |          s is a Hanoi(n disk) seq on a..z
. | |          ⇒
. | |          (∀x:{a...z}, i:{n'+1...n}. s(a,i) = s(x,i))
. | |          ⇒
. | |          s is a Hanoi(n' disk) seq on a..z
. | |    {hanoi_seq_shallower}
. | |
. | |\
. | | ⊢  s is a Hanoi(n disk) seq on y..z
. | |   by BackThru:
. | |      Thm*  ∀n:ℕ, a,z:ℤ, s:({a...z}→{1...n}→Peg).
. | |            s is a Hanoi(n disk) seq on a..z
. | |            ⇒
. | |            (∀a',z':{a...z}. s is a Hanoi(n disk) seq on a'..z')
. | |      {hanoi_subseq}
. | |      Using:[a:= a]
. | |
. | .\
. | . 24. x1 : {y...z}
. | . 25. i : {n-1+1...n}
. | . ⊢  s(y,i) = s(x1,i)
. | .   by Subst: i = n ∈ {1...n} THEN Rewrite by Hyp:18
. |
. |\
. | ⊢  s(z) = (λi.q) ∈ {1...n-1}→Peg  by Inclusion Hyp:11
. |
. .\
. . 24. (2^(n-1))≤z-y+1
. . ⊢  2·(2^(n-1))≤z-a+1
. .   by x+1-y≤0 ∣ (x-a+1)+(z-y+1)≤z-a+1  Asserted
END
```