```
n0n1_irregular  THM

⊢  ∀S:ActionSet(ℤ), s,q:S.car.
|  (∃n:ℕ+ . #(S.car)=n ) & (∀k:ℕ. (S:n0n1(k)←s) = q) ⇒
|  (∃L:ℤ*. (S:L←s) = q & (∀k:ℕ. ¬L = n0n1(k)))  by UnivCD
|
1. S : ActionSet(ℤ)
2. s : S.car
3. q : S.car
4. (∃n:ℕ+ . #(S.car)=n ) & (∀k:ℕ. (S:n0n1(k)←s) = q)
⊢  ∃L:ℤ*. (S:L←s) = q & (∀k:ℕ. ¬L = n0n1(k))
| by Analyze 4 THEN Analyze 4
|
4. n : ℕ+
5. #(S.car)=n
6. ∀k:ℕ. (S:n0n1(k)←s) = q
⊢   by Unfold ‘card‘ 5 THEN Unfold ‘one_one_corr‘ 5
|
5. ∃f:(S.car→ℕn), g:(ℕn→S.car). InvFuns(S.car; ℕn; f; g)
6. ∀k:ℕ. (S:n0n1(k)←s) = q
⊢   by Analyze 5
|
5. f : S.car→ℕn
6. ∃g:(ℕn→S.car). InvFuns(S.car; ℕn; f; g)
7. ∀k:ℕ. (S:n0n1(k)←s) = q
⊢
| by Inst
|    Thm* ∀n:{1...}, m:{n+1...}, f:(ℕm→ℕn).
|         ∃i,j:ℕm. i<j & f(i) = f(j)
|    {phole_lemma}
|    [n;n+1;λi.f((S:([1]↑i)←s))]
|
8. ∃i,j:ℕ(n+1).
|  i<j & (λi.f((S:([1]↑i)←s)))(i) = (λi.f((S:([1]↑i)←s)))(j)
⊢   by Analyze 8 THEN Analyze 9 THEN Analyze 10
|
8. i : ℕ(n+1)
9. j : ℕ(n+1)
10. i<j
11. (λi.f((S:([1]↑i)←s)))(i) = (λi.f((S:([1]↑i)←s)))(j)
⊢   by Reduce 11
|
11. f((S:([1]↑i)←s)) = f((S:([1]↑j)←s))
⊢   by Analyze 6
|
6. g : ℕn→S.car
7. InvFuns(S.car; ℕn; f; g)
8. ∀k:ℕ. (S:n0n1(k)←s) = q
9. i : ℕ(n+1)
10. j : ℕ(n+1)
11. i<j
12. f((S:([1]↑i)←s)) = f((S:([1]↑j)←s))
⊢   by ApplyFunToHypEquands g 12
|
13. g(f((S:([1]↑i)←s))) = g(f((S:([1]↑j)←s)))
```

```
⊢   by RWH add_composeC 13
|
13. (g o f)((S:([1]↑i)←s)) = (g o f)((S:([1]↑j)←s))
⊢   by Unfold `inv_funs` 7
|
7. g o f = Id & f o g = Id
8. ∀k:ℕ. (S:n0n1(k)←s) = q
9. i : ℕ(n+1)
10. j : ℕ(n+1)
11. i<j
12. f((S:([1]↑i)←s)) = f((S:([1]↑j)←s))
13. (g o f)((S:([1]↑i)←s)) = (g o f)((S:([1]↑j)←s))
⊢   by Analyze 7
|
7. g o f = Id
8. f o g = Id
9. ∀k:ℕ. (S:n0n1(k)←s) = q
10. i : ℕ(n+1)
11. j : ℕ(n+1)
12. i<j
13. f((S:([1]↑i)←s)) = f((S:([1]↑j)←s))
14. (g o f)((S:([1]↑i)←s)) = (g o f)((S:([1]↑j)←s))
⊢   by RWH (HypC 7) 14
|
14. Id((S:([1]↑i)←s)) = Id((S:([1]↑j)←s))
⊢   by Reduce 14
|
14. (S:([1]↑i)←s) = (S:([1]↑j)←s)
⊢   by Witness ([0]↑j) @ ([1]↑i)
|
⊢  (S:([0]↑j) @ ([1]↑i)←s) = q
|  & (∀k:ℕ. ¬(([0]↑j) @ ([1]↑i)) = n0n1(k))  by Analyze 0
|
|\
| ⊢  (S:([0]↑j) @ ([1]↑i)←s) = q
| | by Inst
| |    Thm* ∀S:ActionSet(Alph), s:S.car, L1,L2,L:Alph*.
| |         (S:L1←s) = (S:L2←s) ⇒ (S:L @ L1←s) = (S:L @ L2←s)
| |    {maction_lemma}
| |    [ℤ;S;s;([1]↑i);([1]↑j);([0]↑j)]
| |
| |\
| | .....antecedent .....
| | ⊢  (S:([1]↑i)←s) = (S:([1]↑j)←s)  by Auto
| |
| .\
| . 15. (S:([0]↑j) @ ([1]↑i)←s) = (S:([0]↑j) @ ([1]↑j)←s)
| . ⊢   by RWH (HypC 15) 0
| . |
| . ⊢  (S:([0]↑j) @ ([1]↑j)←s) = q  by RWH (FoldC `n0n1`) 0
| . |
| . ⊢  (S:n0n1(j)←s) = q  by Witness9 j
|
.\
. ⊢  ∀k:ℕ. ¬(([0]↑j) @ ([1]↑i)) = n0n1(k)  by Analyze 0
```

```
. |
. 15. k : ℕ
. ⊢  ¬(([0]↑j) @ ([1]↑i)) = n0n1(k)  by Analyze 0
. |
. 16. (([0]↑j) @ ([1]↑i)) = n0n1(k)
. ⊢  False  by Decide (k = i)
. |
. |\
. | 17. k = i
. | ⊢   by RW (UnfoldC `n0n1`) 16
. | |
. | 16. (([0]↑j) @ ([1]↑i)) = (([0]↑k) @ ([1]↑k))
. | 17. k = i
. | ⊢   by ApFunToHypEquands `L` ||0:L|| ℕ 16
. | |
. | 18. ||0:([0]↑j) @ ([1]↑i)|| = ||0:([0]↑k) @ ([1]↑k)||
. | ⊢
. | | by RWH
. | |    (LemmaC
. | |     Thm* ∀e:ℤ, L,M:ℤ*. ||e:L @ M|| = ||e:L||+||e:M|| ∈ ℕ
. | |     {counter_append})
. | |    18
. | |
. | 18. ||0:([0]↑j)||+||0:([1]↑i)|| = ||0:([0]↑k)||+||0:([1]↑k)||
. | ⊢   by Analyze 18
. | |
. | 18. ||0:([0]↑j)||+||0:([1]↑i)|| = ||0:([0]↑k)||+||0:([1]↑k)||
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by RW0 "counter_lpower" 18
. | |
. | 18. j•||0:[0]||+i•||0:[1]|| = k•||0:[0]||+k•||0:[1]||
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by RWH (RecUnfoldC `el_counter`) 18
. | |
. | 18. j•if
. | |      null([0])→0
. | |     ;0=₂hd([0])→1+||0:tl([0])||
. | |     else ||0:tl([0])|| fi
. | |   +i•if
. | |       null([1])→0
. | |      ;0=₂hd([1])→1+||0:tl([1])||
. | |      else ||0:tl([1])|| fi
. | |   =
. | |   k•if
. | |      null([0])→0
. | |     ;0=₂hd([0])→1+||0:tl([0])||
. | |     else ||0:tl([0])|| fi
. | |   +k•if
. | |       null([1])→0
. | |      ;0=₂hd([1])→1+||0:tl([1])||
. | |      else ||0:tl([1])|| fi
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by Reduce 18
. | |
. | 18. j•(1+||0:nil||)+i•||0:nil|| = k•(1+||0:nil||)+k•||0:nil||
```

```
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by RWH (RecUnfoldC ‘el_counter‘) 18
. | |
. | 18. j·(1+if
. | |         null(nil)→0
. | |        ;0=₂hd(nil)→1+||0:tl(nil)||
. | |        else ||0:tl(nil)|| fi)
. | |   +i·if
. | |       null(nil)→0
. | |      ;0=₂hd(nil)→1+||0:tl(nil)||
. | |      else ||0:tl(nil)|| fi
. | |   =
. | |   k·(1+if
. | |         null(nil)→0
. | |        ;0=₂hd(nil)→1+||0:tl(nil)||
. | |        else ||0:tl(nil)|| fi)
. | |   +k·if
. | |       null(nil)→0
. | |      ;0=₂hd(nil)→1+||0:tl(nil)||
. | |      else ||0:tl(nil)|| fi
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by Reduce 18
. | |
. | 18. j·1+i·0 = k·1+k·0
. | 19. ||0:([0]↑j)||+||0:([1]↑i)||≥0
. | ⊢   by Auto
. |
. .\
. . 17. ¬k = i
. . ⊢   by RW (UnfoldC ‘n0n1‘) 16
. . |
. . 16. (([0]↑j) @ ([1]↑i)) = (([0]↑k) @ ([1]↑k))
. . 17. ¬k = i
. . ⊢   by ApFunToHypEquands ‘L’ ||1:L|| ℕ 16
. . |
. . 18. ||1:([0]↑j) @ ([1]↑i)|| = ||1:([0]↑k) @ ([1]↑k)||
. . ⊢

. . | by RWH
. . |    (LemmaC
. . |     Thm* ∀e:ℤ, L,M:ℤ*. ||e:L @ M|| = ||e:L||+||e:M|| ∈ ℕ
. . |     {counter_append})
. . |    18
. . |
. . 18. ||1:([0]↑j)||+||1:([1]↑i)|| = ||1:([0]↑k)||+||1:([1]↑k)||
. . ⊢   by Analyze 18
. . |
. . 18. ||1:([0]↑j)||+||1:([1]↑i)|| = ||1:([0]↑k)||+||1:([1]↑k)||
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by RWO "counter_lpower" 18
. . |
. . 18. j·||1:[0]||+i·||1:[1]|| = k·||1:[0]||+k·||1:[1]||
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by RWH (RecUnfoldC ‘el_counter‘) 18
. . |
. . 18. j·if
```

```
. . |      null([0])→0
. . |     ;1=₂hd([0])→1+||1:tl([0])||
. . |     else ||1:tl([0])|| fi
. . |   +i·if
. . |       null([1])→0
. . |      ;1=₂hd([1])→1+||1:tl([1])||
. . |      else ||1:tl([1])|| fi
. . |   =
. . |   k·if
. . |      null([0])→0
. . |     ;1=₂hd([0])→1+||1:tl([0])||
. . |     else ||1:tl([0])|| fi
. . |   +k·if
. . |       null([1])→0
. . |      ;1=₂hd([1])→1+||1:tl([1])||
. . |      else ||1:tl([1])|| fi
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by Reduce 18
. . |
. . 18. j·||1:nil||+i·(1+||1:nil||) = k·||1:nil||+k·(1+||1:nil||)
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by RWH (RecUnfoldC ‘el_counter‘) 18
. . |
. . 18. j·if
. . |      null(nil)→0
. . |     ;1=₂hd(nil)→1+||1:tl(nil)||
. . |     else ||1:tl(nil)|| fi
. . |   +i·(1+if
. . |          null(nil)→0
. . |         ;1=₂hd(nil)→1+||1:tl(nil)||
. . |         else ||1:tl(nil)|| fi)
. . |   =
. . |   k·if
. . |      null(nil)→0
. . |     ;1=₂hd(nil)→1+||1:tl(nil)||
. . |     else ||1:tl(nil)|| fi
. . |   +k·(1+if
. . |          null(nil)→0
. . |         ;1=₂hd(nil)→1+||1:tl(nil)||
. . |         else ||1:tl(nil)|| fi)
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by Reduce 18
. . |
. . 18. j·0+i·1 = k·0+k·1
. . 19. ||1:([0]↑j)||+||1:([1]↑i)||≥0
. . ⊢   by Auto
END
```