```
refine_iso  THM

⊢  ∀A1:Automata(Alph;S1), A2:Automata(Alph;S2).
|  Con(A1) & Con(A2) ⇒
|  LangOf(A1) = LangOf(A2) ⇒ A1 ⊆ A2  & A2 ⊆ A1  ⇒ A1 ≡ A2  by Auto
|
1. Alph : Type
2. S1 : Type
3. S2 : Type
4. A1 : Automata(Alph;S1)
5. A2 : Automata(Alph;S2)
6. Con(A1)
7. Con(A2)
8. LangOf(A1) = LangOf(A2)
9. A1 ⊆ A2
10. A2 ⊆ A1
⊢  A1 ≡ A2  by Unfold ‘connected‘ 6
|
6. ∀s:S1. ∃l:Alph*. (Result(A1)l) = s
7. Con(A2)
8. LangOf(A1) = LangOf(A2)
9. A1 ⊆ A2
10. A2 ⊆ A1
⊢
| by Inst
|    Thm* ∀Q:(A→B→Prop).
|         (∀x:A. ∃y:B. Q(x,y)) ⇒ (∃f:(A→B). ∀x:A. Q(x,f(x)))
|    {ax_choice}
|    [S1;Alph*;λ2s,l. (Result(A1)l) = s]
|
11. ∃f:(S1→Alph*). ∀x:S1. (Result(A1)f(x)) = x
⊢   by Analyze 11
|
11. f : S1→Alph*
12. ∀x:S1. (Result(A1)f(x)) = x
⊢   by Unfold ‘auto_iso‘ 0
|
⊢  ∃f:(S1→S2).
|  Bij(S1; S2; f)
|  & (∀s:S1, a:Alph. f(δA1(s,a)) = δA2(f(s),a))
|  & f(InitialState(A1)) = InitialState(A2)
|  & (∀s:S1. FinalState(A1)(s) = FinalState(A2)(f(s)))
| by Witness λs.Result(A2)f(s)
|
|\
| ⊢  Bij(S1; S2; λs.Result(A2)f(s))  by Analyze 0
| |
| |\
| | ⊢  Inj(S1; S2; λs.Result(A2)f(s))
| | | by Unfold ‘inject‘ 0 THEN Reduce 0
| | |
| | 13. a1 : S1
| | 14. a2 : S1
| | 15. (Result(A2)f(a1)) = (Result(A2)f(a2))
| | ⊢  a1 = a2  by Unfold ‘refine‘ 10
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| | |
| | 10. ∀x,y:Alph*.
| | |   (Result(A2)x) = (Result(A2)y) ⇒ (Result(A1)x) = (Result(A1)y)
| | 11. f : S1→Alph*
| | 12. ∀x:S1. (Result(A1)f(x)) = x
| | 13. a1 : S1
| | 14. a2 : S1
| | 15. (Result(A2)f(a1)) = (Result(A2)f(a2))
| | ⊢   by InstHyp [f(a1);f(a2)] 10
| | |
| | 16. (Result(A1)f(a1)) = (Result(A1)f(a2))
| | ⊢   by RWH (HypC 12) 16
| |
| .\
| . ⊢  Surj(S1; S2; λs.Result(A2)f(s))
| . | by Unfold `surject` 0 THEN Reduce 0
| . |
| . 13. b : S2
| . ⊢  ∃a:S1. (Result(A2)f(a)) = b  by Unfold `connected` 7
| . |
| . 7. ∀s:S2. ∃l:Alph*. (Result(A2)l) = s
| . 8. LangOf(A1) = LangOf(A2)
| . 9. A1 ⊆ A2
| . 10. A2 ⊆ A1
| . 11. f : S1→Alph*
| . 12. ∀x:S1. (Result(A1)f(x)) = x
| . 13. b : S2
| . ⊢   by Witness7 b
| . |
| . 7. LangOf(A1) = LangOf(A2)
| . 8. A1 ⊆ A2
| . 9. A2 ⊆ A1
| . 10. f : S1→Alph*
| . 11. ∀x:S1. (Result(A1)f(x)) = x
| . 12. b : S2
| . 13. ∃l:Alph*. (Result(A2)l) = b
| . ⊢   by Analyze 13
| . |
| . 13. l : Alph*
| . 14. (Result(A2)l) = b
| . ⊢   by Witness Result(A1)l
| . |
| . ⊢  (Result(A2)f(Result(A1)l)) = b  by Unfold `refine` 8
| . |
| . 8. ∀x,y:Alph*.
| . |  (Result(A1)x) = (Result(A1)y) ⇒ (Result(A2)x) = (Result(A2)y)
| . 9. A2 ⊆ A1
| . 10. f : S1→Alph*
| . 11. ∀x:S1. (Result(A1)f(x)) = x
| . 12. b : S2
| . 13. l : Alph*
| . 14. (Result(A2)l) = b
| . ⊢   by Witness11 Result(A1)l
| . |
| . 11. b : S2
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| . 12. l : Alph*
| . 13. (Result(A2)l) = b
| . 14. (Result(A1)f(Result(A1)l)) = (Result(A1)l)
| . ⊢   by InstHyp [f(Result(A1)l);l] 8
| . |
| . 15. (Result(A2)f(Result(A1)l)) = (Result(A2)l)
| . ⊢   by RWH (HypC 13) 15
|
|\
| 13. s : S1
| 14. a : Alph
| ⊢  (λs.Result(A2)f(s))(δA1(s,a)) = δA2((λs.Result(A2)f(s))(s),a)
| | by Reduce 0
| |
| ⊢  (Result(A2)f(δA1(s,a))) = δA2((Result(A2)f(s)),a)
| | by Assert (δA2((Result(A2)f(s)),a) = (Result(A2)a.f(s)))
| |
| |\
| | .....assertion .....
| | ⊢  δA2((Result(A2)f(s)),a) = (Result(A2)a.f(s))
| | | by RW (AddrC [3] (RecUnfoldC `compute_list`)) 0
| | |
| | ⊢  δA2((Result(A2)f(s)),a)
| | |  =
| | |  if null(a.f(s))→ InitialState(A2)
| | |  else δA2((Result(A2)tl((a.f(s)))),hd((a.f(s)))) fi
| | | by Reduce 0
| | |
| | ⊢  δA2((Result(A2)f(s)),a) = δA2((Result(A2)f(s)),a)  by Auto
| |
| .\
| . 15. δA2((Result(A2)f(s)),a) = (Result(A2)a.f(s))
| . ⊢   by RWH (HypC 15) 0
| . |
| . ⊢  (Result(A2)f(δA1(s,a))) = (Result(A2)a.f(s))
| . | by InstHyp [s] 12
| . |
| . 16. (Result(A1)f(s)) = s
| . ⊢   by RW (AddrC [2;2;2;1;2] (RevHypC 16)) 0
| . |
| . ⊢  (Result(A2)f(δA1((Result(A1)f(s)),a))) = (Result(A2)a.f(s))
| . | by Unfold `refine` 9
| . |
| . 9. ∀x,y:Alph*.
| . |  (Result(A1)x) = (Result(A1)y) ⇒ (Result(A2)x) = (Result(A2)y)
| . 10. A2 ⊆ A1
| . 11. f : S1→Alph*
| . 12. ∀x:S1. (Result(A1)f(x)) = x
| . 13. s : S1
| . 14. a : Alph
| . 15. δA2((Result(A2)f(s)),a) = (Result(A2)a.f(s))
| . 16. (Result(A1)f(s)) = s
| . ⊢   by BackThru 9
| . |
| . ⊢  (Result(A1)f(δA1((Result(A1)f(s)),a))) = (Result(A1)a.f(s))
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| . | by RW (AddrC [2] (HypC 12)) 0
| . |
| . ⊢  δA1((Result(A1)f(s)),a) = (Result(A1)a.f(s))
| .   by RW (AddrC [3] (RecUnfoldC `compute_list`)) 0 THEN Reduce 0
|
|\
| ⊢  (λs.Result(A2)f(s))(InitialState(A1)) = InitialState(A2)
| | by Reduce 0
| |
| ⊢  (Result(A2)f(InitialState(A1))) = InitialState(A2)
| | by Assert (InitialState(A2) = (Result(A2)nil))
| |
| |\
| | .....assertion .....
| | ⊢  InitialState(A2) = (Result(A2)nil)
| |   by RecUnfold `compute_list` 0 THEN Reduce 0
| |
| .\
| . 13. InitialState(A2) = (Result(A2)nil)
| . ⊢   by RWH (HypC 13) 0
| . |
| . ⊢  (Result(A2)f(InitialState(A1))) = (Result(A2)nil)
| . | by Unfold `refine` 9
| . |
| . 9. ∀x,y:Alph*.
| . |  (Result(A1)x) = (Result(A1)y) ⇒ (Result(A2)x) = (Result(A2)y)
| . 10. A2 ⊆ A1
| . 11. f : S1→Alph*
| . 12. ∀x:S1. (Result(A1)f(x)) = x
| . 13. InitialState(A2) = (Result(A2)nil)
| . ⊢   by BackThru 9
| . |
| . ⊢  (Result(A1)f(InitialState(A1))) = (Result(A1)nil)
| . | by RWH (HypC 12) 0
| . |
| . ⊢  InitialState(A1) = (Result(A1)nil)
| .   by RecUnfold `compute_list` 0 THEN Reduce 0

|
.\
. 13. s : S1
. ⊢  FinalState(A1)(s) = FinalState(A2)((λs.Result(A2)f(s))(s))
. | by Reduce 0
. |
. ⊢  FinalState(A1)(s) = FinalState(A2)(Result(A2)f(s))
. | by Unfold `lang_eq` 8
. |
. 8. ∀l:Alph*. LangOf(A1)(l) ⟺ LangOf(A2)(l)
. 9. A1 ⊆ A2
. 10. A2 ⊆ A1
. 11. f : S1→Alph*
. 12. ∀x:S1. (Result(A1)f(x)) = x
. 13. s : S1
. ⊢   by Unfold `auto_lang` 8
. |
. 8. ∀l:Alph*. (λl.A1(l)↓)(l) ⟺ (λl.A2(l)↓)(l)
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. 9. A1 ⊆ A2
. 10. A2 ⊆ A1
. 11. f : S1→Alph*
. 12. ∀x:S1. (Result(A1)f(x)) = x
. 13. s : S1
. ⊢   by Reduce 8
. |
. 8. ∀l:Alph*. A1(l)↓ ⟺ A2(l)↓
. 9. A1 ⊆ A2
. 10. A2 ⊆ A1
. 11. f : S1→Alph*
. 12. ∀x:S1. (Result(A1)f(x)) = x
. 13. s : S1
. ⊢   by Unfold ‘accept_list‘ 8
. |
. 8. ∀l:Alph*.
. |  FinalState(A1)(Result(A1)l) ⟺ FinalState(A2)(Result(A2)l)
. 9. A1 ⊆ A2
. 10. A2 ⊆ A1
. 11. f : S1→Alph*
. 12. ∀x:S1. (Result(A1)f(x)) = x
. 13. s : S1
. ⊢
. | by BackThru
. |    Thm* ∀a,b:𝔹. (a ⟺ b) ⇒ a = b    {iff_imp_equal_bool}
. |
. ⊢  FinalState(A1)(s) ⟺ FinalState(A2)(Result(A2)f(s))
. | by Witness8 f(s)
. |
. 8. A1 ⊆ A2
. 9. A2 ⊆ A1
. 10. f : S1→Alph*
. 11. ∀x:S1. (Result(A1)f(x)) = x
. 12. s : S1
. 13. FinalState(A1)(Result(A1)f(s)) ⟺ FinalState(A2)(Result(A2)f(s))
. ⊢   by RWH (HypC 11) 13
END
```