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list_quo_choice_pls  THM

⊢  ∀q:ℕ+, E:(ℕq*→ℕq*→Prop).
|  (EquivRel x,y:ℕq*. x E y) & (∀x,y:ℕq*. Dec(x E y)) ⇒
|  (∃h:(ℕq*→ℕq*).
|   (∀x,y:ℕq*. (x E y) ⟺ h(x) = h(y)) & (∀x:ℕq*. x E (h(x))))
| by UnivCD THEN Analyze 3
|
1. q : ℕ+
2. E : ℕq*→ℕq*→Prop
3. EquivRel x,y:ℕq*. x E y
4. ∀x,y:ℕq*. Dec(x E y)
⊢  ∃h:(ℕq*→ℕq*).
|  (∀x,y:ℕq*. (x E y) ⟺ h(x) = h(y)) & (∀x:ℕq*. x E (h(x)))
| by Inst Thm* ∀q:ℕ+. (ℕq*) ∼ ℕ    {list_1_1_nat} [q]
|
5. (ℕq*) ∼ ℕ
⊢   by Analyze 5 THEN Analyze 6 THEN Analyze 7
|
5. f : ℕq*→ℕ
6. g : ℕ→ℕq*
7. g o f = Id
8. f o g = Id
⊢
| by Inst
|    Thm* ∀E:(ℕ→ℕ→Prop).
|         (EquivRel x,y:ℕ. x E y) & (∀x,y:ℕ. Dec(x E y)) ⇒
|         (∃h:(ℕ→ℕ).
|          (∀n,k:ℕ. (n E k) ⟺ h(n) = h(k)) & (∀n:ℕ. n E (h(n))))
|    {comp_choice}
|    [λx,y. (g(x)) E (g(y))]
|
|\
| .....antecedent .....
| ⊢  (EquivRel x,y:ℕ. x (λx,y. (g(x)) E (g(y))) y)
| |  & (∀x,y:ℕ. Dec(x (λx,y. (g(x)) E (g(y))) y))
| | by Unfold ‘equiv_rel‘ 0 THEN Reduce 0 THEN Unfold ‘refl‘ 0
| |    THEN
| |    Unfold ‘sym‘ 0
| |    THEN
| |    Unfold ‘trans‘ 0
| |    THEN
| |    Analyze 3
| |    THEN
| |    Analyze 4
| |    THEN
| |    Unfold ‘refl‘ 3
| |    THEN
| |    Unfold ‘sym‘ 4
| |    THEN
| |    Unfold ‘trans‘ 5
| |
| |\
| | 3. ∀a:ℕq*. a E a
| | 4. ∀a,b:ℕq*. (a E b) ⇒ (b E a)
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| | 5. ∀a,b,c:ℕq*. (a E b) ⇒ (b E c) ⇒ (a E c)
| | 6. ∀x,y:ℕq*. Dec(x E y)
| | 7. f : ℕq*→ℕ
| | 8. g : ℕ→ℕq*
| | 9. g o f = Id
| | 10. f o g = Id
| | 11. a : ℕ
| | ⊢  (g(a)) E (g(a))  by BackThru 3
| |
| |\
| | 3. ∀a:ℕq*. a E a
| | 4. ∀a,b:ℕq*. (a E b) ⇒ (b E a)
| | 5. ∀a,b,c:ℕq*. (a E b) ⇒ (b E c) ⇒ (a E c)
| | 6. ∀x,y:ℕq*. Dec(x E y)
| | 7. f : ℕq*→ℕ
| | 8. g : ℕ→ℕq*
| | 9. g o f = Id
| | 10. f o g = Id
| | 11. a : ℕ
| | 12. b : ℕ
| | 13. (g(a)) E (g(b))
| | ⊢  (g(b)) E (g(a))  by FwdThru 4 [13]
| |
| .\
| . 3. ∀a:ℕq*. a E a
| . 4. ∀a,b:ℕq*. (a E b) ⇒ (b E a)
| . 5. ∀a,b,c:ℕq*. (a E b) ⇒ (b E c) ⇒ (a E c)
| . 6. ∀x,y:ℕq*. Dec(x E y)
| . 7. f : ℕq*→ℕ
| . 8. g : ℕ→ℕq*
| . 9. g o f = Id
| . 10. f o g = Id
| . 11. a : ℕ
| . 12. b : ℕ
| . 13. c : ℕ
| . 14. (g(a)) E (g(b))
| . 15. (g(b)) E (g(c))
| . ⊢  (g(a)) E (g(c))  by FwdThru 5 [14;15]
|
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. 9. ∃h:(ℕ→ℕ).
. |  (∀n,k:ℕ. (n (λx,y. (g(x)) E (g(y))) k) ⟺ h(n) = h(k))
. |  & (∀n:ℕ. n (λx,y. (g(x)) E (g(y))) (h(n)))
. ⊢   by Analyze 9 THEN Witness g o h o f
. |
. 9. h : ℕ→ℕ
. 10. (∀n,k:ℕ. (n (λx,y. (g(x)) E (g(y))) k) ⟺ h(n) = h(k))
. |   & (∀n:ℕ. n (λx,y. (g(x)) E (g(y))) (h(n)))
. ⊢  (∀x,y:ℕq*. (x E y) ⟺ (g o h o f)(x) = (g o h o f)(y))
. |  & (∀x:ℕq*. x E ((g o h o f)(x)))  by Reduce 10
. |
. 10. (∀n,k:ℕ. ((g(n)) E (g(k))) ⟺ h(n) = h(k))
. |   & (∀n:ℕ. (g(n)) E (g(h(n))))
. ⊢   by UnivCD
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. ⊢   by Analyze 10 THEN Analyze 0
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. |\
. | 10. ∀n,k:ℕ. ((g(n)) E (g(k))) ⟺ h(n) = h(k)
. | 11. ∀n:ℕ. (g(n)) E (g(h(n)))
. | ⊢  ∀x,y:ℕq*. (x E y) ⟺ (g o h o f)(x) = (g o h o f)(y)
. | | by UnivCD
. | |
. | 12. x : ℕq*
. | 13. y : ℕq*
. | ⊢  (x E y) ⟺ (g o h o f)(x) = (g o h o f)(y)
. | | by InstHyp [f(x);f(y)] 10
. | |
. | 14. ((g(f(x))) E (g(f(y)))) ⟺ h(f(x)) = h(f(y))
. | ⊢   by RWH add_composeC 14
. | |
. | 14. (((g o f)(x)) E ((g o f)(y))) ⟺ (h o f)(x) = (h o f)(y)
. | ⊢   by RWH (HypC 7) 14 THEN Reduce 14
. | |
. | 14. (x E y) ⟺ h(f(x)) = h(f(y))
. | ⊢   by Analyze 0 THEN Analyze 0
. | |
. | |\
. | | 15. x E y
. | | ⊢  (g o h o f)(x) = (g o h o f)(y)  by RWH (HypC 14) 15
. | | |
. | | 15. h(f(x)) = h(f(y))
. | | ⊢   by RWH rem_composeC 0 THEN RWH rem_composeC 0
. | | |
. | | ⊢  g(h(f(x))) = g(h(f(y)))  by RWH (HypC 15) 0
. | |
. | .\
. | . 15. (g o h o f)(x) = (g o h o f)(y)
. | . ⊢  x E y  by RWH (HypC 14) 0
. | . |
. | . ⊢  h(f(x)) = h(f(y))  by ApFunToHypEquands ‘z’ (f(z)) ℕ 15
. | . |
. | . 16. f((g o h o f)(x)) = f((g o h o f)(y))
. | . ⊢   by RWH add_composeC 16
. | . |
. | . 16. (f o g o h o f)(x) = (f o g o h o f)(y)
. | . ⊢
. | . | by RWH
. | . |    (LemmaC
. | . |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. | . |          h o (g o f) = (h o g) o f ∈ A→D
. | . |     {comp_assoc})
. | . |    16
. | . |
. | . 16. (f o g o h o f)(x) = (f o g o h o f)(y)
. | . ⊢   by RWH (HypC 8) 16
. | . |
. | . 16. (Id o h o f)(x) = (Id o h o f)(y)
. | . ⊢   by Reduce 16
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. .\
. . 10. ∀n,k:ℕ. ((g(n)) E (g(k))) ⟺ h(n) = h(k)
. . 11. ∀n:ℕ. (g(n)) E (g(h(n)))
. . ⊢  ∀x:ℕq*. x E ((g o h o f)(x))  by Analyze 0 THEN Witness11 f(x)
. . |
. . 11. x : ℕq*
. . 12. (g(f(x))) E (g(h(f(x))))
. . ⊢  x E ((g o h o f)(x))  by RWH add_composeC 12
. . |
. . 12. ((g o f)(x)) E ((g o h)(f(x)))
. . ⊢   by RWH (HypC 7) 12 THEN Reduce 12
. . |
. . 12. x E (g(h(f(x))))
. . ⊢   by Reduce 0
END
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