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nsub_bij_ooc_invpair  THM

⊢  (ℕa bij→ ℕa) ~ (ℕa←→ℕa)  by Auto
|
1. a : ℕ
⊢  (ℕa bij→ ℕa) ~ (ℕa←→ℕa)
| by Witness:
|      (λf.<f,λy.least x:ℕ. f(x)=₂y>) ∈ (ℕa bij→ ℕa)→ℕa←→ℕa |
|      (λfg.fg.1) ∈ (ℕa←→ℕa)→ℕa bij→ ℕa
|
|\
| .....assertion .....
| ⊢  (λf.<f,λy.least x:ℕ. f(x)=₂y>) ∈ (ℕa bij→ ℕa)→ℕa←→ℕa  by Auto
| |
| 2. f : ℕa bij→ ℕa
| ⊢  <f,λy.least x:ℕ. f(x)=₂y> ∈ ℕa←→ℕa
| | by Guarding (least [var]:ℕ. <bool>) Analyze THEN Reduce Concl
| |
| |\
| | ⊢  <f,λy.least x:ℕ. f(x)=₂y> ∈ (ℕa→ℕa)×(ℕa→ℕa)
| | | by f ∈ ℕa bij→ ℕa  Asserted THEN Analyze-1
| | |
| | 3. f ∈ ℕa→ℕa
| | 4. Bij(ℕa; ℕa; f)
| | ⊢  <f,λy.least x:ℕ. f(x)=₂y> ∈ (ℕa→ℕa)×(ℕa→ℕa)
| | | by Analyze THEN Analyze
| | |
| | 5. y : ℕa
| | ⊢  (least x:ℕ. f(x)=₂y) ∈ ℕa
| |   by BackThru:
| |      Thm*  ∀f:(ℕa onto→ ℕb), y:ℕb. (least x:ℕ. f(x)=₂y) ∈ ℕa
| |      {nsub_surj_least_preimage_total}
| |
| .\
| . ⊢  InvFuns(ℕa;ℕa;f;λy.least x:ℕ. f(x)=₂y)
| .   by BackThru:
| .      Thm*  ∀f:(ℕa bij→ ℕa). InvFuns(ℕa;ℕa;f;λy.least x:ℕ. f(x)=₂y)
| .      {nsub_bij_least_preimage_inverse}
|
|\
| .....assertion .....
| 2. (λf.<f,λy.least x:ℕ. f(x)=₂y>) ∈ (ℕa bij→ ℕa)→ℕa←→ℕa
| ⊢  (λfg.fg.1) ∈ (ℕa←→ℕa)→ℕa bij→ ℕa
| | by Analyze
| |    THEN
| |    Analyze-1 THEN New:[f | g] Analyze-2 THEN OnAllClauses Reduce
| |    THEN
| |    Analyze
| |
| .....set predicate .....
| 3. f : ℕa→ℕa
| 4. g : ℕa→ℕa
| 5. InvFuns(ℕa;ℕa;f;g)
| ⊢  Bij(ℕa; ℕa; f)
|   by FwdThru:
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|       Thm*  InvFuns(A;B;f;g) ⇒ Bij(A; B; f)
|       {fun_with_inv2_is_bij}
|       on [ InvFuns(ℕa;ℕa;f;g) ]
|
.\
. 2. (λf.<f,λy.least x:ℕ. f(x)=_2y>) ∈ (ℕa bij→ ℕa)→ℕa←→ℕa
. 3. (λfg.fg.1) ∈ (ℕa←→ℕa)→ℕa bij→ ℕa
. ⊢  InvFuns(ℕa bij→ ℕa;ℕa←→ℕa;λf.<f,λy.least x:ℕ. f(x)=_2y>;λfg.fg.1)
. | by Analyze
. |
. |\
. | ⊢  ∀x:(ℕa bij→ ℕa).
. | |  (λfg.fg.1)((λf.<f,λy.least x:ℕ. f(x)=_2y>)(x)) = x
. | | by New:f Analyze
. | |
. | 4. f : ℕa bij→ ℕa
. | ⊢  (λfg.fg.1)((λf.<f,λy.least x:ℕ. f(x)=_2y>)(f)) = f
. |   by Reduce Concl
. |
. .\
. . ⊢  ∀y:(ℕa←→ℕa). (λf.<f,λy.least x:ℕ. f(x)=_2y>)((λfg.fg.1)(y)) = y
. . | by Analyze THEN Analyze-1 THEN New:[f | g] Analyze-2
. . |    THEN
. . |    Reduce Hyp:-1
. . |
. . 4. f : ℕa→ℕa
. . 5. g : ℕa→ℕa
. . 6. InvFuns(ℕa;ℕa;f;g)
. . ⊢  (λf.<f,λy.least x:ℕ. f(x)=_2y>)((λfg.fg.1)(<f,g>))
. . |  =
. . |  <f,g>
. . |  ∈ ℕa←→ℕa  by Reduce Concl
. . |
. . ⊢  <f,λy.least x:ℕ. f(x)=_2y> = <f,g> ∈ ℕa←→ℕa
. . | by f ∈ ℕa onto→ ℕa |
. . |    (λy.least x:ℕ. f(x)=_2y) ∈ ℕa→ℕa |
. . |    InvFuns(ℕa;ℕa;f;λy.least x:ℕ. f(x)=_2y)
. . |    Asserted
. . |
. . |\
. . | .....assertion .....
. . | ⊢  f ∈ ℕa onto→ ℕa  by Analyze
. . | |
. . | .....set predicate .....
. . | ⊢  Surj(ℕa; ℕa; f)
. . |   by FwdThru:
. . |      Thm*  InvFuns(A;B;f;g) ⇒ Surj(A; B; f)
. . |      {fun_with_inv2_is_surj}
. . |      on [ InvFuns(ℕa;ℕa;f;g) ]
. . |
. . |\
. . | .....assertion .....
. . | 7. f ∈ ℕa onto→ ℕa
. . | ⊢  (λy.least x:ℕ. f(x)=_2y) ∈ ℕa→ℕa
. . |   by Analyze
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. . |      THEN
. . |      BackThru:
. . |      Thm*  ∀f:(ℕa onto→ ℕb), y:ℕb. (least x:ℕ. f(x)=2y) ∈ ℕa
. . |      {nsub_surj_least_preimage_total}
. . |
. . |\
. . | .....assertion .....
. . | 7. f ∈ ℕa onto→ ℕa
. . | 8. (λy.least x:ℕ. f(x)=2y) ∈ ℕa→ℕa
. . | ⊢  InvFuns(ℕa;ℕa;f;λy.least x:ℕ. f(x)=2y)
. . | | by Def THEN Reduce Concl
. . | |
. . | |\
. . | | 9. x : ℕa
. . | | ⊢  (least x@0:ℕ. f(x@0)=2f(x)) = x
. . | | | by Use:[b:= a | y:= f(x)]
. . | | |    Inst:
. . | | |    Thm*  ∀f:(ℕa onto→ ℕb), y:ℕb. f(least x:ℕ. f(x)=2y) = y
. . | | |    {nsub_surj_least_preimage_works}
. . | | |
. . | | 10. f(least x@0:ℕ. f(x@0)=2f(x)) = f(x)
. . | | ⊢  (least x@0:ℕ. f(x@0)=2f(x)) = x  by ApFun: Hyp:8 to: f(x)
. . | | |
. . | | 11. (least x@0:ℕ. f(x@0)=2f(x)) ∈ ℕa
. . | | ⊢  (least x@0:ℕ. f(x@0)=2f(x)) = x
. . | | | by BackThru: Inj(ℕa; ℕa; f)
. . | | |
. . | | .....assertion .....
. . | | ⊢  Inj(ℕa; ℕa; f)
. . | |   by FwdThru:
. . | |      Thm*  InvFuns(A;B;f;g) ⇒ Inj(A; B; f)
. . | |      {fun_with_inv2_is_inj}
. . | |      on [ InvFuns(ℕa;ℕa;f;g) ]
. . | |
. . | .\
. . | . 9. y : ℕa
. . | . ⊢  f(least x:ℕ. f(x)=2y) = y
. . | .   by BackThru:
. . | .      Thm*  ∀f:(ℕa onto→ ℕb), y:ℕb. f(least x:ℕ. f(x)=2y) = y
. . | .      {nsub_surj_least_preimage_works}
. . |
. . .\
. . . 7. f ∈ ℕa onto→ ℕa
. . . 8. (λy.least x:ℕ. f(x)=2y) ∈ ℕa→ℕa
. . . 9. InvFuns(ℕa;ℕa;f;λy.least x:ℕ. f(x)=2y)
. . . ⊢  <f,λy.least x:ℕ. f(x)=2y> = <f,g> ∈ ℕa←→ℕa
. . . | by FwdThru:
. . . |    Thm*  InvFuns(A;B;f;g) ⇒ InvFuns(A;B;f;h) ⇒ g = h
. . . |    {inv_funs_2_unique}
. . . |    on [ Hyp:6 | Hyp:9 ]
. . . |
. . . 10. g = (λy.least x:ℕ. f(x)=2y)
. . . ⊢  <f,λy.least x:ℕ. f(x)=2y> = <f,g> ∈ ℕa←→ℕa
. . .   by Repeat (Analyze THEN Reduce Concl)
END
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