```
auto2_lemma_3  THM

⊢  ∀R:(Alph*→Alph*→Prop), n:ℕ+.
|  (∀x:Alph*. R(x,x))
|  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
|  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
|  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
|  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
|  ⇒
|
|  (∀a,b,c:Alph*.
|   ||a||≥n·n ⇒
|   (∃a':Alph*. ||a'||<||a|| & R((a @ b),a' @ b) & R((a @ c),a' @ c)))
| by GenUnivCD THEN Analyze 4 THEN Analyze 5 THEN Analyze 6
|    THEN
|    Analyze 7
|    THEN
|    Analyze 8
|
1. Alph : Type
2. R : Alph*→Alph*→Prop
3. n : ℕ+
4. ∀x:Alph*. R(x,x)
5. ∀x,y:Alph*. R(x,y) ⇒ R(y,x)
6. ∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z)
7. ∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y)
8. w : ℕn→Alph*
9. ∀l:Alph*. ∃i:ℕn. R(l,w(i))
10. a : Alph*
11. b : Alph*
12. c : Alph*
13. ||a||≥n·n
⊢  ∃a':Alph*. ||a'||<||a|| & R((a @ b),a' @ b) & R((a @ c),a' @ c)
| by Inst
|    Thm* ∀Q:(A→B→Prop).
|          (∀x:A. ∃y:B. Q(x,y)) ⇒ (∃f:(A→B). ∀x:A. Q(x,f(x)))
|    {ax_choice}
|    [Alph*;ℕn;λ2l,i. R(l,w(i))]
|    THEN
|    Analyze 14
|
14. f : Alph*→ℕn
15. ∀x:Alph*. R(x,w(f(x)))
⊢
| by Inst
|    Thm* ∀n,m:ℕ+. ∃f:(ℕn×ℕm→ℕ(n·m)). Bij(ℕn×ℕm; ℕ(n·m); f)
|    {rect_enumer}
|    [n;n]
|    THEN
|    Analyze 16
|
16. f1 : ℕn×ℕn→ℕ(n·n)
17. Bij(ℕn×ℕn; ℕ(n·n); f1)
⊢
| by Inst
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|    Thm* ∀n:{1...}, m:{n+1...}, f:(ℕm→ℕn).
|         ∃i,j:ℕm. i<j & f(i) = f(j)
|    {phole_lemma}
|    [n·n
|    ;n·n+1
|    ;λj.f1
|        (<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)]
|
|\
| .....set predicate .....
| ⊢  1≤n·n
| | by Inst Thm* ∀a,b:ℤ, n:ℕ. a≤b ⇒ n·a≤n·b    {mul_preserves_le}
| |    [1;n;n]
| |
| 18. n·1≤n·n
| ⊢   by RWH (RevLemmaC Thm* ∀i:ℤ. i = i·1    {mul_ident}) 18
|
.\
. 18. ∃i,j:ℕ(n·n+1).
. |   i<j
. |   & (λj.f1
. |         (<f((a[||a||-j..||a||⁻]) @ b)
. |          ,f((a[||a||-j..||a||⁻]) @ c)>))
. |     (i)
. |     =
. |     (λj.f1
. |         (<f((a[||a||-j..||a||⁻]) @ b)
. |          ,f((a[||a||-j..||a||⁻]) @ c)>))
. |     (j)
. ⊢   by Analyze 18 THEN Analyze 19 THEN Analyze 20 THEN Reduce 21
. |
. 18. i : ℕ(n·n+1)
. 19. j : ℕ(n·n+1)
. 20. i<j
. 21. f1(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. |   =
. |   f1(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. ⊢   by Witness (a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])
. |
. ⊢  ||(a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])||<||a||
. |  & R((a @ b),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ b)
. |  & R((a @ c),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ c)
. | by Analyze 0
. |
. |\
. | ⊢  ||(a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])||<||a||
. |   by RWH
. |      (LemmaC
. |       Thm* ∀as,bs:T*. ||as @ bs|| = ||as||+||bs||
. |       {length_append})
. |      0
. |      THEN
. |      RWH
. |      (LemmaC
. |       Thm* ∀as:T*, i:{0...||as||}, j:{i...||as||}.
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. |            ||as[i..j⁻]|| = j-i
. |       {length_segment})
. |      0
. |
. .\
. . ⊢  R((a @ b),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ b)
. . |  & R((a @ c),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ c)
. . | by Unfold ‘biject‘ 17
. . |
. . 17. Inj(ℕn×ℕn; ℕ(n·n); f1) & Surj(ℕn×ℕn; ℕ(n·n); f1)
. . 18. i : ℕ(n·n+1)
. . 19. j : ℕ(n·n+1)
. . 20. i<j
. . 21. f1(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. . |   =
. . |   f1(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. . ⊢   by Analyze 17 THEN Unfold ‘inject‘ 17
. . |
. . 17. ∀a1,a2:(ℕn×ℕn). f1(a1) = f1(a2) ⇒ a1 = a2
. . 18. Surj(ℕn×ℕn; ℕ(n·n); f1)
. . 19. i : ℕ(n·n+1)
. . 20. j : ℕ(n·n+1)
. . 21. i<j
. . 22. f1(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. . |   =
. . |   f1(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. . ⊢
. . | by Witness17
. . |      <f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>
. . |    THEN
. . |    Witness22
. . |      <f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>
. . |    THEN
. . |    Analyze 22
. . |
. . 17. Surj(ℕn×ℕn; ℕ(n·n); f1)
. . 18. i : ℕ(n·n+1)
. . 19. j : ℕ(n·n+1)
. . 20. i<j
. . 21. f1(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. . |   =
. . |   f1(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. . 22. <f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>
. . |   =
. . |   <f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>
. . ⊢   by Analyze 22
. . |
. . 22. 1of(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. . |   =
. . |   1of(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. . 23. 2of(<f((a[||a||-i..||a||⁻]) @ b),f((a[||a||-i..||a||⁻]) @ c)>)
. . |   =
. . |   2of(<f((a[||a||-j..||a||⁻]) @ b),f((a[||a||-j..||a||⁻]) @ c)>)
. . ⊢   by Reduce 22 THEN Reduce 23
. . |
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. . 22. f((a[||a||-i..||a||⁻]) @ b) = f((a[||a||-j..||a||⁻]) @ b)
. . 23. f((a[||a||-i..||a||⁻]) @ c) = f((a[||a||-j..||a||⁻]) @ c)
. . ⊢
. . | by InstHyp [(a[||a||-i..||a||⁻]) @ b] 15
. . |    THEN
. . |    InstHyp [(a[||a||-j..||a||⁻]) @ b] 15
. . |    THEN
. . |    InstHyp [(a[||a||-i..||a||⁻]) @ c] 15
. . |    THEN
. . |    InstHyp [(a[||a||-j..||a||⁻]) @ c] 15
. . |
. . 24. R(((a[||a||-i..||a||⁻]) @ b),w(f((a[||a||-i..||a||⁻]) @ b)))
. . 25. R(((a[||a||-j..||a||⁻]) @ b),w(f((a[||a||-j..||a||⁻]) @ b)))
. . 26. R(((a[||a||-i..||a||⁻]) @ c),w(f((a[||a||-i..||a||⁻]) @ c)))
. . 27. R(((a[||a||-j..||a||⁻]) @ c),w(f((a[||a||-j..||a||⁻]) @ c)))
. . ⊢   by RWH (HypC 22) 24 THEN RWH (HypC 23) 26
. . |
. . 24. R(((a[||a||-i..||a||⁻]) @ b),w(f((a[||a||-j..||a||⁻]) @ b)))
. . 25. R(((a[||a||-j..||a||⁻]) @ b),w(f((a[||a||-j..||a||⁻]) @ b)))
. . 26. R(((a[||a||-i..||a||⁻]) @ c),w(f((a[||a||-j..||a||⁻]) @ c)))
. . 27. R(((a[||a||-j..||a||⁻]) @ c),w(f((a[||a||-j..||a||⁻]) @ c)))
. . ⊢
. . | by InstHyp
. . |    [(a[||a||-i..||a||⁻]) @ b;w(f((a[||a||-j..||a||⁻]) @ b))]
. . |    5
. . |    THEN
. . |    InstHyp
. . |    [(a[||a||-i..||a||⁻]) @ c;w(f((a[||a||-j..||a||⁻]) @ c))]
. . |    5
. . |
. . 28. R(w(f((a[||a||-j..||a||⁻]) @ b)),(a[||a||-i..||a||⁻]) @ b)
. . 29. R(w(f((a[||a||-j..||a||⁻]) @ c)),(a[||a||-i..||a||⁻]) @ c)
. . ⊢
. . | by InstHyp
. . |    [(a[||a||-j..||a||⁻]) @ b
. . |    ;w(f((a[||a||-j..||a||⁻]) @ b))
. . |    ;(a[||a||-i..||a||⁻]) @ b]
. . |    6
. . |    THEN
. . |    InstHyp
. . |    [(a[||a||-j..||a||⁻]) @ c
. . |    ;w(f((a[||a||-j..||a||⁻]) @ c))
. . |    ;(a[||a||-i..||a||⁻]) @ c]
. . |    6
. . |
. . 30. R(((a[||a||-j..||a||⁻]) @ b),(a[||a||-i..||a||⁻]) @ b)
. . 31. R(((a[||a||-j..||a||⁻]) @ c),(a[||a||-i..||a||⁻]) @ c)
. . ⊢
. . | by InstHyp
. . |    [(a[||a||-j..||a||⁻]) @ b
. . |    ;(a[||a||-i..||a||⁻]) @ b
. . |    ;a[0..||a||-j⁻]]
. . |    7
. . |    THEN
. . |    InstHyp
```

```
. . |    [(a[||a||-j..||a||⁻]) @ c
. . |    ;(a[||a||-i..||a||⁻]) @ c
. . |    ;a[0..||a||-j⁻]]
. . |    7
. . |
. . 32. R
. . |   (((a[0..||a||-j⁻]) @ (a[||a||-j..||a||⁻]) @ b)
. . |   ,(a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻]) @ b)
. . 33. R
. . |   (((a[0..||a||-j⁻]) @ (a[||a||-j..||a||⁻]) @ c)
. . |   ,(a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻]) @ c)
. . ⊢
. . | by RWH
. . |    (RevLemmaC
. . |     Thm* ∀as,bs,cs:T*. ((as @ bs) @ cs) = (as @ (bs @ cs))
. . |     {append_assoc})
. . |    32
. . |    THEN
. . |    RWH
. . |    (RevLemmaC
. . |     Thm* ∀as,bs,cs:T*. ((as @ bs) @ cs) = (as @ (bs @ cs))
. . |     {append_assoc})
. . |    33
. . |
. . 32. R
. . |   ((((a[0..||a||-j⁻]) @ (a[||a||-j..||a||⁻])) @ b)
. . |   ,((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ b)
. . 33. R
. . |   ((((a[0..||a||-j⁻]) @ (a[||a||-j..||a||⁻])) @ c)
. . |   ,((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ c)
. . ⊢
. . | by RWH
. . |    (LemmaC
. . |     Thm* ∀as:T*, i:{0...||as||}, j:{i...||as||}, k:{j...||as||}.
. . |          ((as[i..j⁻]) @ (as[j..k⁻])) = (as[i..k⁻])
. . |     {append_segment})
. . |    32
. . |    THEN
. . |    RWH
. . |    (LemmaC
. . |     Thm* ∀as:T*, i:{0...||as||}, j:{i...||as||}, k:{j...||as||}.
. . |          ((as[i..j⁻]) @ (as[j..k⁻])) = (as[i..k⁻])
. . |     {append_segment})
. . |    33
. . |
. . 32. R
. . |   (((a[0..||a||⁻]) @ b)
. . |   ,((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ b)
. . 33. R
. . |   (((a[0..||a||⁻]) @ c)
. . |   ,((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ c)
. . ⊢
. . | by RWH
. . |    (LemmaC
. . |     Thm* ∀as:T*. (as[0..||as||⁻]) = as    {whole_segment})
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. . |     32
. . |     THEN
. . |     RWH
. . |     (LemmaC
. . |      Thm* ∀as:T*. (as[0..||as||⁻]) = as    {whole_segment})
. . |     33
. . |
. . 32. R((a @ b),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ b)
. . 33. R((a @ c),((a[0..||a||-j⁻]) @ (a[||a||-i..||a||⁻])) @ c)
. . ⊢   by Analyze 0
END
```