```
auto2_lemma_4  THM

⊢  ∀R:(Alph*→Alph*→Prop), n:ℕ+ .
|  (∀x:Alph*. R(x,x))
|  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
|  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
|  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
|  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
|  ⇒
|
|  (∀a,b,c:Alph*.
|   ∃a':Alph*. ||a'||<n•n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| by GenUnivCD
|
1. Alph : Type
2. R : Alph*→Alph*→Prop
3. n : ℕ+
4. (∀x:Alph*. R(x,x))
|  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
|  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
|  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
|  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
5. a : Alph*
6. b : Alph*
7. c : Alph*
⊢  ∃a':Alph*. ||a'||<n•n & R((a @ b),a' @ b) & R((a @ c),a' @ c)
| by Inst
|    Thm* ∀P:(Alph*→Prop).
|         (∀n:ℕ.
|          (∀l:Alph*. ||l||<n ⇒ P(l)) ⇒
|          (∀l:Alph*. ||l|| = n ⇒ P(l)))
|         ⇒
|         (∀l:Alph*. P(l))
|    {auto2_list_ind}
|    []
|
8. ∀Alph:Type, P:(Alph*→Prop).
|  (∀n:ℕ.
|   (∀l:Alph*. ||l||<n ⇒ P(l)) ⇒ (∀l:Alph*. ||l|| = n ⇒ P(l)))
|  ⇒
|  (∀l:Alph*. P(l))
⊢   by Witness8 Alph
|
8. ∀P:(Alph*→Prop).
|  (∀n:ℕ.
|   (∀l:Alph*. ||l||<n ⇒ P(l)) ⇒ (∀l:Alph*. ||l|| = n ⇒ P(l)))
|  ⇒
|  (∀l:Alph*. P(l))
⊢
| by Witness8
|      λa.∃a':Alph*.
|      ||a'||<n•n & R((a @ b),a' @ b) & R((a @ c),a' @ c)
|
8. (∀n@0:ℕ.
|   (∀l:Alph*.
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|    ||l||<n@0 ⇒
|    (λa.∃a':Alph*.
|     ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
|    (l))
|   ⇒
|
|   (∀l:Alph*.
|    ||l|| = n@0 ⇒
|    (λa.∃a':Alph*.
|     ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
|    (l)))
|  ⇒
|
|  (∀l:Alph*.
|   (λa.∃a':Alph*. ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
|   (l))
⊢   by Analyze 8
|
|\
| .....antecedent .....
| 7. c : Alph*
| ⊢  ∀n@0:ℕ.
| |  (∀l:Alph*.
| |   ||l||<n@0 ⇒
| |   (λa.∃a':Alph*.
| |    ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| |   (l))
| |  ⇒
| |
| |  (∀l:Alph*.
| |   ||l|| = n@0 ⇒
| |   (λa.∃a':Alph*.
| |    ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| |   (l))  by Thin 5
| |
| 5. b : Alph*
| 6. c : Alph*
| ⊢   by Analyze 0
| |
| 7. n@0 : ℕ
| ⊢  (∀l:Alph*.
| |   ||l||<n@0 ⇒
| |   (λa.∃a':Alph*.
| |    ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| |   (l))
| |  ⇒
| |

| |  (∀l:Alph*.
| |   ||l|| = n@0 ⇒
| |   (λa.∃a':Alph*.
| |    ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| |   (l))  by Reduce 0 THEN UnivCD
| |
| 8. ∀l:Alph*.
| |  ||l||<n@0 ⇒
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| |  (∃a':Alph*. ||a'||<n·n & R((l @ b),a' @ b) & R((l @ c),a' @ c))
| 9. l : Alph*
| 10. ||l|| = n@0
| ⊢  ∃a':Alph*. ||a'||<n·n & R((l @ b),a' @ b) & R((l @ c),a' @ c)
| | by Inst
| |    Thm* ∀R:(Alph*→Alph*→Prop), n:ℕ⁺.
| |         (∀x:Alph*. R(x,x))
| |         & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
| |         & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| |         & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| |         & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| |         ⇒
| |
| |         (∀a,b,c:Alph*.
| |          ||a||≥n·n ⇒
| |          (∃a':Alph*.
| |           ||a'||<||a|| & R((a @ b),a' @ b) & R((a @ c),a' @ c)))
| |    {auto2_lemma_3}
| |    [Alph;R;n]
| |
| 11. ∀a,b,c:Alph*.
| |   ||a||≥n·n ⇒
| |   (∃a':Alph*.
| |    ||a'||<||a|| & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| ⊢   by Decide (||l||<n·n)
| |
| |\
| | 12. ||l||<n·n
| | ⊢   by Witness l
| | |
| | ⊢  ||l||<n·n & R((l @ b),l @ b) & R((l @ c),l @ c)  by Analyze 0
| | |
| | |\
| | | ⊢  ||l||<n·n  by Auto
| | |
| | .\
| | . ⊢  R((l @ b),l @ b) & R((l @ c),l @ c)  by Analyze 4
| | . |
| | . 4. ∀x:Alph*. R(x,x)
| | . 5. (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
| | . |  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| | . |  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | . |  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| | . 6. b : Alph*
| | . 7. c : Alph*
| | . 8. n@0 : ℕ
| | . 9. ∀l:Alph*.
| | . |  ||l||<n@0 ⇒
| | . |  (∃a':Alph*.
| | . |   ||a'||<n·n & R((l @ b),a' @ b) & R((l @ c),a' @ c))
| | . 10. l : Alph*
| | . 11. ||l|| = n@0
| | . 12. ∀a,b,c:Alph*.
| | . |   ||a||≥n·n ⇒
| | . |   (∃a':Alph*.
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| | . |    ||a'||<||a|| & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| | . 13. ||l||<n•n
| | . ⊢   by Analyze 0
| | . |
| | . |\
| | . | ⊢  R((l @ b),l @ b)  by Witness4 l @ b
| | . |
| | . .\
| | . . ⊢  R((l @ c),l @ c)  by Witness4 l @ c
| |
| .\
| . 12. ¬||l||<n•n
| . ⊢   by Witness11 l THEN Witness12 b
| . |
| . 11. ¬||l||<n•n
| . 12. ∀c:Alph*.
| . |   ||l||≥n•n ⇒
| . |   (∃a':Alph*.
| . |    ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c))
| . ⊢   by Witness12 c
| . |
| . 12. ||l||≥n•n ⇒
| . |   (∃a':Alph*.
| . |    ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c))
| . ⊢   by Analyze 12
| . |
| . 12. ∃a':Alph*.
| . |   ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c)
| . ⊢   by Analyze 12
| . |
| . 12. a' : Alph*
| . 13. ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c)
| . ⊢   by Witness8 a'
| . |
| . 8. l : Alph*
| . 9. ||l|| = n@0
| . 10. ¬||l||<n•n
| . 11. a' : Alph*
| . 12. ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c)
| . 13. ||a'||<n@0 ⇒
| . |   (∃a'@0:Alph*.
| . |    ||a'@0||<n•n & R((a' @ b),a'@0 @ b) & R((a' @ c),a'@0 @ c))
| . ⊢   by Analyze 13
| . |
| . |\
| . | .....antecedent .....
| . | 12. ||a'||<||l|| & R((l @ b),a' @ b) & R((l @ c),a' @ c)
| . | ⊢  ||a'||<n@0  by Analyze 12
| . |
| . .\
| . . 13. ∃a'@0:Alph*.
| . . |   ||a'@0||<n•n & R((a' @ b),a'@0 @ b) & R((a' @ c),a'@0 @ c)
| . . ⊢   by Analyze 13
| . . |
| . . 13. a'@0 : Alph*
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| . . 14. ||a'@0||<n·n & R((a' @ b),a'@0 @ b) & R((a' @ c),a'@0 @ c)
| . . ⊢   by Witness a'@0
| . . |
| . . ⊢  ||a'@0||<n·n & R((l @ b),a'@0 @ b) & R((l @ c),a'@0 @ c)
| . . | by Analyze 14
| . . |
| . . 14. ||a'@0||<n·n
| . . 15. R((a' @ b),a'@0 @ b) & R((a' @ c),a'@0 @ c)
| . . ⊢   by Analyze 0
| . . |
| . . |\
| . . | ⊢  ||a'@0||<n·n  by Auto
| . . |
| . . .\
| . . . ⊢  R((l @ b),a'@0 @ b) & R((l @ c),a'@0 @ c)
| . . . | by Analyze 4 THEN Analyze 5 THEN Analyze 6 THEN Analyze 7
| . . . |    THEN
| . . . |    Analyze 16
| . . . |    THEN
| . . . |    Analyze 17
| . . . |    THEN
| . . . |    Analyze 21
| . . . |    THEN
| . . . |    Analyze 0
| . . . |
| . . . |\
| . . . | 4. ∀x:Alph*. R(x,x)
| . . . | 5. ∀x,y:Alph*. R(x,y) ⇒ R(y,x)
| . . . | 6. ∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z)
| . . . | 7. ∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y)
| . . . | 8. ∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i))
| . . . | 9. b : Alph*
| . . . | 10. c : Alph*
| . . . | 11. n@0 : ℕ
| . . . | 12. l : Alph*
| . . . | 13. ||l|| = n@0
| . . . | 14. ¬||l||<n·n
| . . . | 15. a' : Alph*
| . . . | 16. ||a'||<||l||
| . . . | 17. R((l @ b),a' @ b)
| . . . | 18. R((l @ c),a' @ c)
| . . . | 19. a'@0 : Alph*
| . . . | 20. ||a'@0||<n·n
| . . . | 21. R((a' @ b),a'@0 @ b)
| . . . | 22. R((a' @ c),a'@0 @ c)
| . . . | ⊢  R((l @ b),a'@0 @ b)
| . . . |   by InstHyp [l @ b;a' @ b;a'@0 @ b] 6
| . . . |
| . . . .\
| . . . . 4. ∀x:Alph*. R(x,x)
| . . . . 5. ∀x,y:Alph*. R(x,y) ⇒ R(y,x)
| . . . . 6. ∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z)
| . . . . 7. ∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y)
| . . . . 8. ∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i))
| . . . . 9. b : Alph*
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| . . . . 10. c : Alph*
| . . . . 11. n@0 : ℕ
| . . . . 12. l : Alph*
| . . . . 13. ||l|| = n@0
| . . . . 14. ¬||l||<n·n
| . . . . 15. a' : Alph*
| . . . . 16. ||a'||<||l||
| . . . . 17. R((l @ b),a' @ b)
| . . . . 18. R((l @ c),a' @ c)
| . . . . 19. a'@0 : Alph*
| . . . . 20. ||a'@0||<n·n
| . . . . 21. R((a' @ b),a'@0 @ b)
| . . . . 22. R((a' @ c),a'@0 @ c)
| . . . . ⊢  R((l @ c),a'@0 @ c)
| . . . .   by InstHyp [l @ c;a' @ c;a'@0 @ c] 6
|
.\
. 8. ∀l:Alph*.
. |  (λa.∃a':Alph*.
. |   ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
. |  (l)
. ⊢   by Witness8 a
. |
. 8. (λa.∃a':Alph*.
.     ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
.    (a)
. ⊢   by Reduce 8

END
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