```
auto2_lemma_5  THM

⊢  ∀n:ℕ. Fin(Alph) ⇒ Fin({l:(Alph*)| ||l|| = n })
| by Unfold ‘finite‘ 0 THEN UnivCD
|
1. Alph : Type
2. n : ℕ
3. ∃n:ℕ, f:(ℕn→Alph). Bij(ℕn; Alph; f)
⊢  ∃n@0:ℕ, f:(ℕn@0→{l:(Alph*)| ||l|| = n }).
|  Bij(ℕn@0; {l:(Alph*)| ||l|| = n }; f)  by Analyze 3 THEN Analyze 4
|
3. n1 : ℕ
4. f : ℕn1→Alph
5. Bij(ℕn1; Alph; f)
⊢   by Witness (n1↑n)
|
⊢  ∃f:(ℕ(n1↑n)→{l:(Alph*)| ||l|| = n }).
|  Bij(ℕ(n1↑n); {l:(Alph*)| ||l|| = n }; f)
| by Inst
|    Thm* ∀n,m:ℕ. ∃f:((ℕm→ℕn)→ℕ(n↑m)). Bij(ℕm→ℕn; ℕ(n↑m); f)
|    {fun_enumer}
|    [n1;n]
|
6. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)). Bij(ℕn→ℕn1; ℕ(n1↑n); f)
⊢
| by Inst
|    Thm* (∃f:(A→B). Bij(A; B; f)) ⟺ (A ∼ B)    {bij_iff_1_1_corr}
|    [ℕn→ℕn1;ℕ(n1↑n)]
|    THEN
|    Analyze 7
|    THEN
|    Thin 8
|    THEN
|    Analyze 7
|
7. (ℕn→ℕn1) ∼ ℕ(n1↑n)
⊢   by Analyze 7 THEN Analyze 8
|
7. f1 : (ℕn→ℕn1)→ℕ(n1↑n)
8. g : ℕ(n1↑n)→ℕn→ℕn1
9. InvFuns(ℕn→ℕn1; ℕ(n1↑n); f1; g)
⊢   by Witness λz.(f o g(z))[ℕn]
|
|\
| 10. z : ℕ(n1↑n)
| ⊢  ((f o g(z))[ℕn]) ∈ {l:(Alph*)| ||l|| = n }  by Analyze
| |
| |\
| | ⊢  ((f o g(z))[ℕn]) ∈ Alph*  by Auto
| |
| .\
| . .....set predicate .....
| . ⊢  ||(f o g(z))[ℕn]|| = n
| . | by Inst
| . |    Thm* ∀m,n:ℤ, f:({m..n⁻}→T). n<m ∨ ||listify(f; m; n)|| = n-m
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| . |    {listify_length}
| . |    [Alph;0;n;f o g(z)]
| . |
| . 11. n<0 ∨ ||(f o g(z))[ℕn]|| = n-0
| . ⊢   by Analyze 11
|
.\
. ⊢  Bij(ℕ(n1↑n); {l:(Alph*)| ||l|| = n }; λz.(f o g(z))[ℕn])
. | by Analyze 0
. |
. |\
. | ⊢  Inj(ℕ(n1↑n); {l:(Alph*)| ||l|| = n }; λz.(f o g(z))[ℕn])
. | | by Analyze 0
. | |
. | 10. a1 : ℕ(n1↑n)
. | ⊢  ∀a2:ℕ(n1↑n).
. | |  (λz.(f o g(z))[ℕn])(a1) = (λz.(f o g(z))[ℕn])(a2) ⇒ a1 = a2
. | | by Analyze 0
. | |
. | 11. a2 : ℕ(n1↑n)
. | ⊢  (λz.(f o g(z))[ℕn])(a1) = (λz.(f o g(z))[ℕn])(a2) ⇒ a1 = a2
. | | by Reduce 0 THEN Analyze 0
. | |
. | |\
. | | 12. ((f o g(a1))[ℕn]) = ((f o g(a2))[ℕn])
. | | ⊢  a1 = a2  by ApFunToHypEquands ‘x’ (λi.x[i]) (ℕn→Alph) 12
. | | |
. | | |\
. | | | 13. x : {l:(Alph*)| ||l|| = n }
. | | | 14. i : ℕn
. | | | ⊢  i<||x||  by Analyze 13
. | | |
. | | .\
. | | . 13. (λi.((f o g(a1))[ℕn])[i]) = (λi.((f o g(a2))[ℕn])[i])
. | | . ⊢
. | | . | by RWH
. | | . |    (LemmaC
. | | . |     Thm* ∀n:ℕ, f:(ℕn→T), i:ℕn. ((f)[ℕn])[i] = f(i)
. | | . |     {select_listify_id})
. | | . |    13
. | | . |
. | | . 13. (λi.(f o g(a1))(i)) = (λi.(f o g(a2))(i))
. | | . ⊢   by Analyze 5 THEN Thin 6
. | | . |
. | | . 5. Inj(ℕn1; Alph; f)
. | | . 6. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)). Bij(ℕn→ℕn1; ℕ(n1↑n); f)
. | | . 7. f1 : (ℕn→ℕn1)→ℕ(n1↑n)
. | | . 8. g : ℕ(n1↑n)→ℕn→ℕn1
. | | . 9. InvFuns(ℕn→ℕn1; ℕ(n1↑n); f1; g)
. | | . 10. a1 : ℕ(n1↑n)
. | | . 11. a2 : ℕ(n1↑n)
. | | . 12. ((f o g(a1))[ℕn]) = ((f o g(a2))[ℕn])
. | | . 13. (λi.(f o g(a1))(i)) = (λi.(f o g(a2))(i))
. | | . ⊢   by Unfold ‘inject‘ 5
. | | . |
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. | | . 5. ∀a1,a2:ℕn1. f(a1) = f(a2) ⇒ a1 = a2
. | | . 6. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)). Bij(ℕn→ℕn1; ℕ(n1↑n); f)
. | | . 7. f1 : (ℕn→ℕn1)→ℕ(n1↑n)
. | | . 8. g : ℕ(n1↑n)→ℕn→ℕn1
. | | . 9. InvFuns(ℕn→ℕn1; ℕ(n1↑n); f1; g)
. | | . 10. a1 : ℕ(n1↑n)
. | | . 11. a2 : ℕ(n1↑n)
. | | . 12. ((f o g(a1))[ℕn]) = ((f o g(a2))[ℕn])
. | | . 13. (λi.(f o g(a1))(i)) = (λi.(f o g(a2))(i))
. | | . ⊢   by Assert (∀i:ℕn. (f o g(a1))(i) = (f o g(a2))(i))
. | | . |
. | | . |\
. | | . | .....assertion .....
. | | . | ⊢  ∀i:ℕn. (f o g(a1))(i) = (f o g(a2))(i)  by Analyze 0
. | | . | |
. | | . | 14. i : ℕn
. | | . | ⊢  (f o g(a1))(i) = (f o g(a2))(i)
. | | . | | by ApFunToHypEquands ‘x’ (x(i)) Alph 13 THEN Reduce 15
. | | . | |
. | | . | 15. f(g(a1,i)) = f(g(a2,i))
. | | . | ⊢   by Reduce 0
. | | . |
. | | . .\
. | | . . 14. ∀i:ℕn. (f o g(a1))(i) = (f o g(a2))(i)
. | | . . ⊢   by Reduce 14
. | | . . |
. | | . . 14. ∀i:ℕn. f(g(a1,i)) = f(g(a2,i))
. | | . . ⊢   by Assert (∀i:ℕn. g(a1,i) = g(a2,i))
. | | . . |
. | | . . |\
. | | . . | .....assertion .....
. | | . . | ⊢  ∀i:ℕn. g(a1,i) = g(a2,i)  by Analyze 0
. | | . . | |
. | | . . | 15. i : ℕn
. | | . . | ⊢  g(a1,i) = g(a2,i)  by Witness14 i
. | | . . | |
. | | . . | 14. i : ℕn
. | | . . | 15. f(g(a1,i)) = f(g(a2,i))
. | | . . | ⊢   by BackThru 5
. | | . . |
. | | . . .\
. | | . . . 15. ∀i:ℕn. g(a1,i) = g(a2,i)
. | | . . . ⊢   by Analyze 9
. | | . . . |
. | | . . . 9. g o f1 = Id
. | | . . . 10. f1 o g = Id
. | | . . . 11. a1 : ℕ(n1↑n)
. | | . . . 12. a2 : ℕ(n1↑n)
. | | . . . 13. ((f o g(a1))[ℕn]) = ((f o g(a2))[ℕn])
. | | . . . 14. (λi.(f o g(a1))(i)) = (λi.(f o g(a2))(i))
. | | . . . 15. ∀i:ℕn. f(g(a1,i)) = f(g(a2,i))
. | | . . . 16. ∀i:ℕn. g(a1,i) = g(a2,i)
. | | . . . ⊢   by Assert (g(a1) = g(a2))
. | | . . . |
. | | . . . |\
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. | | . . . | .....assertion .....
. | | . . . | ⊢  g(a1) = g(a2)  by Ext
. | | . . . | |
. | | . . . | 17. x : ℕn
. | | . . . | ⊢  g(a1,x) = g(a2,x)  by Witness16 x
. | | . . . |
. | | . . . .\
. | | . . . . 17. g(a1) = g(a2)
. | | . . . . ⊢   by ApFunToHypEquands ‘x’ (f1(x)) ℕ(n1↑n) 17
. | | . . . . |
. | | . . . . 18. f1(g(a1)) = f1(g(a2))
. | | . . . . ⊢   by RWH add_composeC 18
. | | . . . . |
. | | . . . . 18. (f1 o g)(a1) = (f1 o g)(a2)
. | | . . . . ⊢   by RWH (HypC 10) 18
. | | . . . . |
. | | . . . . 18. Id(a1) = Id(a2)
. | | . . . . ⊢   by Reduce 18
. | |
. | |\
. | | ⊢  ((f o g(a1))[ℕn]) ∈ {l:(Alph*)| ||l|| = n }  by Analyze
. | |
. | .\
. | . ⊢  ((f o g(a2))[ℕn]) ∈ {l:(Alph*)| ||l|| = n }  by Analyze
. |
. .\
. . ⊢  Surj(ℕ(n1↑n); {l:(Alph*)| ||l|| = n }; λz.(f o g(z))[ℕn])
. . | by Analyze 0
. . |
. . 10. b : {l:(Alph*)| ||l|| = n }
. . ⊢  ∃a:ℕ(n1↑n). (λz.(f o g(z))[ℕn])(a) = b  by Reduce 0
. . |
. . ⊢  ∃a:ℕ(n1↑n). ((f o g(a))[ℕn]) = b
. . | by Assert (∃a:ℕ(n1↑n). f o g(a) = (λz:ℕ||b||. b[z]))
. . |
. . |\
. . | .....assertion .....
. . | ⊢  ∃a:ℕ(n1↑n). f o g(a) = (λz:ℕ||b||. b[z])
. . | | by Inst
. . | |    Thm* ∀f:(A→B).
. . | |         Bij(A; B; f) ⇒ (∃g:(B→A). InvFuns(A; B; f; g))
. . | |    {bij_imp_exists_inv}
. . | |    [ℕn1;Alph;f]
. . | |
. . | 11. ∃g:(Alph→ℕn1). InvFuns(ℕn1; Alph; f; g)
. . | ⊢   by Analyze 11
. . | |
. . | 11. g1 : Alph→ℕn1
. . | 12. InvFuns(ℕn1; Alph; f; g1)
. . | ⊢   by Assert (∃a:ℕ(n1↑n). g(a) = g1 o (λz:ℕ||b||. b[z]))
. . | |
. . | |\
. . | | .....assertion .....
. . | | ⊢  ∃a:ℕ(n1↑n). g(a) = g1 o (λz:ℕ||b||. b[z])
. . | | | by Analyze 10 THEN Witness f1(g1 o (λz:ℕ||b||. b[z]))
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. . | | |
. . | | 10. b : Alph*
. . | | 11. ||b|| = n
. . | | 12. g1 : Alph→ℕn1
. . | | 13. InvFuns(ℕn1; Alph; f; g1)
. . | | ⊢  g(f1(g1 o (λz:ℕ||b||. b[z]))) = g1 o (λz:ℕ||b||. b[z])
. . | | | by Analyze 9
. . | | |
. . | | 9. g o f1 = Id
. . | | 10. f1 o g = Id
. . | | 11. b : Alph*
. . | | 12. ||b|| = n
. . | | 13. g1 : Alph→ℕn1
. . | | 14. InvFuns(ℕn1; Alph; f; g1)
. . | | ⊢   by RWH add_composeC 0
. . | | |
. . | | ⊢  (g o f1)(g1 o (λz:ℕ||b||. b[z])) = g1 o (λz:ℕ||b||. b[z])
. . | | | by RWH (HypC 9) 0
. . | | |
. . | | ⊢  Id(g1 o (λz:ℕ||b||. b[z])) = g1 o (λz:ℕ||b||. b[z])
. . | | | by Unfold ‘tidentity‘ 0
. . | | |
. . | | ⊢  Id(g1 o (λz:ℕ||b||. b[z])) = g1 o (λz:ℕ||b||. b[z])
. . | | | by Unfold ‘identity‘ 0
. . | | |
. . | | ⊢  (λx.x)(g1 o (λz:ℕ||b||. b[z])) = g1 o (λz:ℕ||b||. b[z])
. . | |   by Reduce 0
. . | |
. . | .\
. . | . 13. ∃a:ℕ(n1↑n). g(a) = g1 o (λz:ℕ||b||. b[z])
. . | . ⊢   by Analyze 13
. . | . |
. . | . 13. a : ℕ(n1↑n)
. . | . 14. g(a) = g1 o (λz:ℕ||b||. b[z])
. . | . ⊢   by ApFunToHypEquands ‘x’ (f o x) (ℕn→Alph) 14
. . | . |
. . | . 15. f o g(a) = f o g1 o (λz:ℕ||b||. b[z])
. . | . ⊢   by Witness a
. . | . |
. . | . |\
. . | . | ⊢  f o g(a) = (λz:ℕ||b||. b[z])
. . | . | | by RWH
. . | . | |    (LemmaC
. . | . | |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. . | . | |          h o (g o f) = (h o g) o f ∈ A→D
. . | . | |     {comp_assoc})
. . | . | |    15
. . | . | |
. . | . | |\
. . | . | | 16. z : ℕn
. . | . | | ⊢  z<||b||  by Analyze 10
. . | . | |
. . | . | .\
. . | . | . 15. f o g(a) = f o g1 o (λz:ℕ||b||. b[z])
. . | . | . ⊢   by Analyze 12
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. . | . | . |
. . | . | . 12. g1 o f = Id
. . | . | . 13. f o g1 = Id
. . | . | . 14. a : ℕ(n1↑n)
. . | . | . 15. g(a) = g1 o (λz:ℕ||b||. b[z])
. . | . | . 16. f o g(a) = f o g1 o (λz:ℕ||b||. b[z])
. . | . | . ⊢   by RWH (HypC 13) 16
. . | . | . |
. . | . | . |\
. . | . | . | 17. z : ℕn
. . | . | . | ⊢  z<||b||  by Analyze 10
. . | . | . |
. . | . | . .\
. . | . | . . 16. f o g(a) = Id o (λz:ℕ||b||. b[z])
. . | . | . . ⊢
. . | . | . . | by RWH
. . | . | . . |    (LemmaC Thm* ∀f:(A→B). Id o f = f    {comp_id_l})
. . | . | . . |    16
. . | . | . . |
. . | . | . . 17. z : ℕn
. . | . | . . ⊢  z<||b||  by Analyze 10
. . | . |
. . | . .\
. . | . . 16. ℕ(n1↑n)
. . | . . 17. z : ℕn
. . | . . ⊢  z<||b||  by Analyze 10
. . |

. . .\
. . . 11. ∃a:ℕ(n1↑n). f o g(a) = (λz:ℕ||b||. b[z])
. . . ⊢   by Analyze 11
. . . |
. . . 11. a : ℕ(n1↑n)
. . . 12. f o g(a) = (λz:ℕ||b||. b[z])
. . . ⊢   by ApFunToHypEquands ‘x’ ((x)[ℕ||b||]) (Alph*) 12
. . . |
. . . |\
. . . | 13. x : ℕn→Alph
. . . | ⊢  x ∈ ℕ||b||→Alph  by Analyze 10
. . . | |
. . . | 10. b : Alph*
. . . | 11. ||b|| = n
. . . | 12. a : ℕ(n1↑n)
. . . | 13. f o g(a) = (λz:ℕ||b||. b[z])
. . . | 14. x : ℕn→Alph
. . . | ⊢   by Auto
. . . |
. . . .\
. . . . 13. ((f o g(a))[ℕ||b||]) = ((λz:ℕ||b||. b[z])[ℕ||b||])
. . . . ⊢
. . . . | by RWH
. . . . |    (LemmaC
. . . . |     Thm* ∀as:T*. ((λi:ℕ||as||. as[i])[ℕ||as||]) = as
. . . . |     {listify_select_id})
. . . . |    13
. . . . |
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. . . . 13. ((f o g(a))[ℕ||b||]) = b
. . . . ⊢   by Analyze 10
. . . . |
. . . . 10. b : Alph*
. . . . 11. ||b|| = n  [not for witness]
. . . . 12. a : ℕ(n1↑n)
. . . . 13. f o g(a) = (λz:ℕ||b||. b[z])
. . . . 14. ((f o g(a))[ℕ||b||]) = b
. . . . ⊢   by Witness a
. . . . |
. . . . |\
. . . . | 11. ||b|| = n
. . . . | 12. a : ℕ(n1↑n)
. . . . | 13. f o g(a) = (λz:ℕ||b||. b[z])
. . . . | 14. ((f o g(a))[ℕ||b||]) = b
. . . . | ⊢  ((f o g(a))[ℕn]) = b  by RWH (HypC 11) 14
. . . . | |
. . . . | 14. ((f o g(a))[ℕn]) = b
. . . . | ⊢   by Analyze
. . . . | |
. . . . | |\
. . . . | | ⊢  ((f o g(a))[ℕn]) = b  by Auto
. . . . | |
. . . . | |\
. . . . | | .....set predicate .....
. . . . | | ⊢  ||(f o g(a))[ℕn]|| = n  by Auto
. . . . | |
. . . . | .\
. . . . | . .....wf .....
. . . . | . 15. l : Alph*
. . . . | . ⊢  (||l|| = n) = (||l|| = n)  by Auto
. . . . |
. . . . .\
. . . . . 11. ||b|| = n
. . . . . 12. a : ℕ(n1↑n)
. . . . . 13. f o g(a) = (λz:ℕ||b||. b[z])
. . . . . 14. ((f o g(a))[ℕ||b||]) = b
. . . . . 15. a1 : ℕ(n1↑n)
. . . . . ⊢  ((f o g(a1))[ℕn]) ∈ {l:(Alph*)| ||l|| = n }  by Analyze
END
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