```
auto2_lemma_7  THM

⊢  ∀R:(Alph*→Alph*→Prop), n:ℕ+, L:(Alph*→𝔹), m:ℕ.
|  (∀x:Alph*. R(x,x))
|  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
|  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
|  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
|  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
|  & (∃v:(ℕm→Alph*). ∀l:Alph*. L(l) ⟺ (∃i:ℕm. R(l,v(i))))
|  & Fin(Alph)
|  ⇒
|  (∀x,y:Alph*. Dec(∃l:Alph*. ¬L(l @ x) = L(l @ y)))  by UnivCD
|
1. Alph : Type
2. R : Alph*→Alph*→Prop
3. n : ℕ+
4. L : Alph*→𝔹
5. m : ℕ
6. (∀x:Alph*. R(x,x))
|  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
|  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
|  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
|  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
|  & (∃v:(ℕm→Alph*). ∀l:Alph*. L(l) ⟺ (∃i:ℕm. R(l,v(i))))
|  & Fin(Alph)
7. x : Alph*
8. y : Alph*
⊢  Dec(∃l:Alph*. ¬L(l @ x) = L(l @ y))
| by Analyze 6 THEN Analyze 7 THEN Analyze 8 THEN Analyze 9
|    THEN
|    Analyze 10
|    THEN
|    Analyze 11
|
6. ∀x:Alph*. R(x,x)
7. ∀x,y:Alph*. R(x,y) ⇒ R(y,x)
8. ∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z)
9. ∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y)
10. ∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i))
11. ∃v:(ℕm→Alph*). ∀l:Alph*. L(l) ⟺ (∃i:ℕm. R(l,v(i)))
12. Fin(Alph)
13. x : Alph*
14. y : Alph*
⊢
| by Assert
|    ((∃l:Alph*. ¬L(l @ x) = L(l @ y))
|     ⟺
|     (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y)))
|
|\
| .....assertion .....
| ⊢  (∃l:Alph*. ¬L(l @ x) = L(l @ y))
| |  ⟺
| |  (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
| | by Analyze 0
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| |
| |\
| | ⊢  (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇒
| | |  (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
| | | by Analyze 0 THEN Analyze 15
| | |
| | 15. l : Alph*
| | 16. ¬L(l @ x) = L(l @ y)
| | ⊢  ∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y)
| | | by Inst
| | |    Thm* ∀R:(Alph*→Alph*→Prop), n:ℕ+.
| | |         (∀x:Alph*. R(x,x))
| | |         & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
| | |         & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| | |         & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | |         & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| | |         ⇒
| | |
| | |         (∀a,b,c:Alph*.
| | |          ∃a':Alph*.
| | |          ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c))
| | |    {auto2_lemma_4}
| | |    [Alph;R;n]
| | |
| | |\
| | | .....antecedent .....
| | | ⊢  (∀x:Alph*. R(x,x))
| | | |  & (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
| | | |  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| | | |  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | | |  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))  by Analyze 0
| | | |
| | | |\
| | | | ⊢  ∀x:Alph*. R(x,x)  by Auto
| | | |
| | | .\
| | | . ⊢  (∀x,y:Alph*. R(x,y) ⇒ R(y,x))
| | | . |  & (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| | | . |  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | | . |  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| | | . | by Analyze 0
| | | . |
| | | . |\
| | | . | ⊢  ∀x,y:Alph*. R(x,y) ⇒ R(y,x)  by Auto
| | | . |
| | | . .\
| | | . . ⊢  (∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z))
| | | . . |  & (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | | . . |  & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| | | . . | by Analyze 0
| | | . . |
| | | . . |\
| | | . . | ⊢  ∀x,y,z:Alph*. R(x,y) & R(y,z) ⇒ R(x,z)  by Auto
| | | . . |
| | | . . .\
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```
| | | . . . ⊢  (∀x,y,z:Alph*. R(x,y) ⇒ R((z @ x),z @ y))
| | | . . .    & (∃w:(ℕn→Alph*). ∀l:Alph*. ∃i:ℕn. R(l,w(i)))
| | | . . .   by Analyze 0
| | |
| | .\
| | . 17. ∀a,b,c:Alph*.
| | . |   ∃a':Alph*.
| | . |   ||a'||<n·n & R((a @ b),a' @ b) & R((a @ c),a' @ c)
| | . ⊢
| | . | by Witness17 l THEN Witness17 x THEN Witness17 y
| | . |    THEN
| | . |    Analyze 17
| | . |    THEN
| | . |    Analyze 18
| | . |    THEN
| | . |    Analyze 19
| | . |
| | . 17. a' : Alph*
| | . 18. ||a'||<n·n
| | . 19. R((l @ x),a' @ x)
| | . 20. R((l @ y),a' @ y)
| | . ⊢   by Witness ||a'|| THEN Witness a'
| | . |
| | . |\
| | . | .....wf .....
| | . | ⊢  a' ∈ {l:(Alph*)| ||l|| = ||a'|| }  by Analyze
| | . |
| | . .\
| | . . ⊢  ¬L(a' @ x) = L(a' @ y)
| | . . | by Assert (∀p,q:Alph*. R(p,q) ⇒ L(p) = L(q))
| | . . |
| | . . |\
| | . . | .....assertion .....
| | . . | ⊢  ∀p,q:Alph*. R(p,q) ⇒ L(p) = L(q)  by UnivCD
| | . . | |
| | . . | 21. p : Alph*
| | . . | 22. q : Alph*
| | . . | 23. R(p,q)
| | . . | ⊢  L(p) = L(q)  by Analyze 11
| | . . | |
| | . . | 11. v : ℕm→Alph*
| | . . | 12. ∀l:Alph*. L(l) ⟺ (∃i:ℕm. R(l,v(i)))
| | . . | 13. Fin(Alph)
| | . . | 14. x : Alph*
| | . . | 15. y : Alph*
| | . . | 16. l : Alph*
| | . . | 17. ¬L(l @ x) = L(l @ y)
| | . . | 18. a' : Alph*
| | . . | 19. ||a'||<n·n
| | . . | 20. R((l @ x),a' @ x)
| | . . | 21. R((l @ y),a' @ y)
| | . . | 22. p : Alph*
| | . . | 23. q : Alph*
| | . . | 24. R(p,q)
| | . . | ⊢   by InstHyp [p] 12 THEN InstHyp [q] 12
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| | . . | |
| | . . | 25. L(p) ⟺ (∃i:ℕm. R(p,v(i)))
| | . . | 26. L(q) ⟺ (∃i:ℕm. R(q,v(i)))
| | . . | ⊢   by Analyze 25 THEN Analyze 27
| | . . | |
| | . . | 25. L(p) ⇒ (∃i:ℕm. R(p,v(i)))
| | . . | 26. (L(p)) ⇐ (∃i:ℕm. R(p,v(i)))
| | . . | 27. L(q) ⇒ (∃i:ℕm. R(q,v(i)))
| | . . | 28. (L(q)) ⇐ (∃i:ℕm. R(q,v(i)))
| | . . | ⊢   by Decide (L(p))
| | . . | |
| | . . | |\
| | . . | | 29. L(p)
| | . . | | ⊢   by Analyze 25
| | . . | | |
| | . . | | 25. (L(p)) ⇐ (∃i:ℕm. R(p,v(i)))
| | . . | | 26. L(q) ⇒ (∃i:ℕm. R(q,v(i)))
| | . . | | 27. (L(q)) ⇐ (∃i:ℕm. R(q,v(i)))
| | . . | | 28. L(p)
| | . . | | 29. ∃i:ℕm. R(p,v(i))
| | . . | | ⊢   by Analyze 29
| | . . | | |
| | . . | | 29. i : ℕm
| | . . | | 30. R(p,v(i))
| | . . | | ⊢   by InstHyp [p;q] 7
| | . . | | |
| | . . | | 31. R(q,p)
| | . . | | ⊢   by InstHyp [q;p;v(i)] 8
| | . . | | |
| | . . | | 32. R(q,v(i))
| | . . | | ⊢   by Analyze 27
| | . . | | |
| | . . | | |\
| | . . | | | .....antecedent .....
| | . . | | | 27. L(p)
| | . . | | | 28. i : ℕm
| | . . | | | 29. R(p,v(i))
| | . . | | | 30. R(q,p)
| | . . | | | 31. R(q,v(i))
| | . . | | | ⊢  ∃i:ℕm. R(q,v(i))  by Witness i
| | . . | | |
| | . . | | .\
| | . . | | . 27. L(p)
| | . . | | . 28. i : ℕm
| | . . | | . 29. R(p,v(i))
| | . . | | . 30. R(q,p)
| | . . | | . 31. R(q,v(i))
| | . . | | . 32. L(q)
| | . . | | . ⊢
| | . . | | . | by Inst Thm* ∀b:𝔹. b ⇒ b = true₂    {assert_elim}
| | . . | | . |    [L(p)]
| | . . | | . |    THEN
| | . . | | . |    Inst Thm* ∀b:𝔹. b ⇒ b = true₂    {assert_elim}
| | . . | | . |    [L(q)]
| | . . | | . |
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| | . . | | . 33. L(p) = true₂
| | . . | | . 34. L(q) = true₂
| | . . | | . ⊢   by Auto
| | . . | |
| | . . | .\
| | . . | . 29. ¬L(p)
| | . . | . ⊢   by Decide (L(q))
| | . . | . |
| | . . | . |\
| | . . | . | 30. L(q)
| | . . | . | ⊢   by Analyze 27
| | . . | . | |
| | . . | . | 27. (L(q)) ⇐ (∃i:ℕm. R(q,v(i)))
| | . . | . | 28. ¬L(p)
| | . . | . | 29. L(q)
| | . . | . | 30. ∃i:ℕm. R(q,v(i))
| | . . | . | ⊢   by Analyze 30
| | . . | . | |
| | . . | . | 30. i : ℕm
| | . . | . | 31. R(q,v(i))
| | . . | . | ⊢   by InstHyp [p;q;v(i)] 8
| | . . | . | |
| | . . | . | 32. R(p,v(i))
| | . . | . | ⊢   by Analyze 26
| | . . | . | |
| | . . | . | |\
| | . . | . | | .....antecedent .....
| | . . | . | | 26. (L(q)) ⇐ (∃i:ℕm. R(q,v(i)))
| | . . | . | | 27. ¬L(p)
| | . . | . | | 28. L(q)
| | . . | . | | 29. i : ℕm
| | . . | . | | 30. R(q,v(i))
| | . . | . | | 31. R(p,v(i))
| | . . | . | | ⊢  ∃i:ℕm. R(p,v(i))  by Witness i
| | . . | . | |
| | . . | . | .\
| | . . | . | . 26. (L(q)) ⇐ (∃i:ℕm. R(q,v(i)))
| | . . | . | . 27. ¬L(p)
| | . . | . | . 28. L(q)
| | . . | . | . 29. i : ℕm
| | . . | . | . 30. R(q,v(i))
| | . . | . | . 31. R(p,v(i))
| | . . | . | . 32. L(p)
| | . . | . | . ⊢   by Auto
| | . . | . |
| | . . | . .\
| | . . | . . 30. ¬L(q)
| | . . | . . ⊢
| | . . | . .   by Inst
| | . . | . .      Thm* ∀b:𝔹. ¬b ⇒ b = false₂    {not_assert_elim}
| | . . | . .      [L(p)]
| | . . | . .      THEN
| | . . | . .      Inst
| | . . | . .      Thm* ∀b:𝔹. ¬b ⇒ b = false₂    {not_assert_elim}
| | . . | . .      [L(q)]
```

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| | . . |
| | . . .\
| | . . . 21. ∀p,q:Alph*. R(p,q) ⇒ L(p) = L(q)
| | . . . ⊢
| | . . . | by InstHyp [l @ x;a’ @ x] 21
| | . . . |    THEN
| | . . . |    InstHyp [l @ y;a’ @ y] 21
| | . . . |
| | . . . 22. L(l @ x) = L(a’ @ x)
| | . . . 23. L(l @ y) = L(a’ @ y)
| | . . . ⊢   by Analyze 0 THEN Analyze 16
| |
| .\
| . ⊢  (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇐ (∃k:ℕ(n·n)
| . |  , l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
| . | by Analyze 0
| . |
| . 15. ∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y)
| . ⊢  ∃l:Alph*. ¬L(l @ x) = L(l @ y)  by Analyze 15
| . |
| . 15. k : ℕ(n·n)
| . 16. ∃l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y)
| . ⊢   by Analyze 16
| . |
| . 16. l : {l:(Alph*)| ||l|| = k }
| . 17. ¬L(l @ x) = L(l @ y)
| . ⊢   by Witness l

|
.\
. 15. (∃l:Alph*. ¬L(l @ x) = L(l @ y))
. |   ⟺
. |   (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. ⊢   by RWH (HypC 15) 0
. |
. ⊢  Dec(∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. | by Inst
. |    Thm* ∀R:(T→Prop).
. |         Fin(T) & (∀t:T. Dec(R(t))) ⇒ Dec(∃t:T. R(t))
. |    {auto2_lemma_6}
. |    [ℕ(n·n);λk.∃l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y)]
. |
. |\
. | 15. (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇒
. | |   (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. | 16. (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇐ (∃k:ℕ(n·n)
. | |   , l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. | ⊢  Fin(ℕ(n·n))
. | | by Inst Thm* ∀n:ℕ. Fin(ℕn)    {nsub_is_finite} [n·n]
. | |
. | .....wf .....
. | ⊢  n·n ∈ ℕ
. |   by Inst Thm* ∀a,b:ℤ, n:ℕ. a≤b ⇒ n·a≤n·b    {mul_preserves_le}
. |      [0;n;n]
. |
. |\
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. | .....decidable? .....
. | 15. (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇒
. | |   (∃k:ℕ(n·n), l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. | 16. (∃l:Alph*. ¬L(l @ x) = L(l @ y)) ⇐ (∃k:ℕ(n·n)
. | |   , l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))
. | 17. t : ℕ(n·n)
. | ⊢  Dec((λk.∃l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))(t))
. | | by Reduce 0
. | |
. | ⊢  Dec(∃l:{l:(Alph*)| ||l|| = t }. ¬L(l @ x) = L(l @ y))
. | | by Inst
. | |    Thm* ∀R:(T→Prop).
. | |         Fin(T) & (∀t:T. Dec(R(t))) ⇒ Dec(∃t:T. R(t))
. | |    {auto2_lemma_6}
. | |    [{l:(Alph*)| ||l|| = t };λl.¬L(l @ x) = L(l @ y)]
. | |
. | |\
. | | ⊢  Fin({l:(Alph*)| ||l|| = t })
. | | | by Inst
. | | |    Thm* ∀n:ℕ. Fin(Alph) ⇒ Fin({l:(Alph*)| ||l|| = n })
. | | |    {auto2_lemma_5}
. | | |    [Alph;t]
. | | |
. | | 18. Fin({l:(Alph*)| ||l|| = t })
. | | ⊢   by Assert (∀l:Alph*. ||l|| = t ⇒ ||l|| = t)
. | | |
. | | |\
. | | | .....assertion .....
. | | | ⊢  ∀l:Alph*. ||l|| = t ⇒ ||l|| = t  by Auto
. | | |
. | | .\
. | | . 19. ∀l:Alph*. ||l|| = t ⇒ ||l|| = t
. | | . ⊢
. | | . | by Inst
. | | . |    Thm* (T ~ U) & Fin(T) ⇒ Fin(U)
. | | . |    {one2one_preserves_fin}
. | | . |    [{l:(Alph*)| ||l|| = t };{l:(Alph*)| ||l|| = t }]
. | | . |
. | | . |\
. | | . | ⊢  {l:(Alph*)| ||l|| = t } ~ {l:(Alph*)| ||l|| = t }
. | | . | | by Inst
. | | . | |    Thm* ∀P,Q:(T→Prop).
. | | . | |         (∀t:T. P(t) ⟺ Q(t)) ⇒
. | | . | |         ({t:T| P(t) } ~ {t:T| Q(t) })
. | | . | |    {auto2_lemma_1}
. | | . | |    [Alph*;λl.||l|| = t;λl.||l|| = t]
. | | . | |
. | | . | |\
. | | . | | .....antecedent .....
. | | . | | ⊢  ∀t@0:Alph*. (λl.||l|| = t)(t@0) ⟺ (λl.||l|| = t)(t@0)
. | | . | |   by Reduce 0
. | | . | |
. | | . | .\
. | | . | . 20. {t@0:(Alph*)| (λl.||l|| = t)(t@0) }
. | | . | .     ~
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. | | . | .     {t@0:(Alph*)| (λl.||l|| = t)(t@0) }
. | | . | . ⊢   by Reduce 20
. | | . |
. | | . .\
. | | . . 20. Fin({l:(Alph*)| ||l|| = t })
. | | . . ⊢   by Auto
. | |
. | |\
. | | .....decidable? .....
. | | 18. t1 : {l:(Alph*)| ||l|| = t }
. | | ⊢  Dec((λl.¬L(l @ x) = L(l @ y))(t1))  by Reduce 0
. | | |
. | | ⊢  Dec(¬L(t1 @ x) = L(t1 @ y))  by ProveDecidable
. | |
. | .\
. | . 18. Dec(∃t:{l:(Alph*)| ||l|| = t }.
. | .     (λl.¬L(l @ x) = L(l @ y))(t))

. | . ⊢   by Reduce 18
. |
. .\
. . 16. Dec(∃t:ℕ(n·n).
. .     (λk.∃l:{l:(Alph*)| ||l|| = k }. ¬L(l @ x) = L(l @ y))(t))
. . ⊢   by Reduce 16
END
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