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fun_preserves_fin  THM

⊢  Fin(S) & Fin(T) ⇒ Fin(S→T)  by Unfold ‘finite‘ 0
|
⊢  ∀S,T:Type.
|  (∃n:ℕ, f:(ℕn→S). Bij(ℕn; S; f)) & (∃n:ℕ, f:(ℕn→T). Bij(ℕn; T; f))
|  ⇒
|  (∃n:ℕ, f:(ℕn→S→T). Bij(ℕn; S→T; f))
| by RWH
|    (LemmaC
|     Thm* (∃f:(A→B). Bij(A; B; f)) ⟺ (A ∼ B)    {bij_iff_1_1_corr})
|    0
|
⊢  ∀S,T:Type. (∃n:ℕ. ℕn ∼ S) & (∃n:ℕ. ℕn ∼ T) ⇒ (∃n:ℕ. ℕn ∼ (S→T))
| by RWH (UnfoldC ‘one_one_corr‘) 0
|
⊢  ∀S,T:Type.
|  (∃n:ℕ, f:(ℕn→S), g:(S→ℕn). InvFuns(ℕn; S; f; g))
|  & (∃n:ℕ, f:(ℕn→T), g:(T→ℕn). InvFuns(ℕn; T; f; g))
|  ⇒
|  (∃n:ℕ, f:(ℕn→S→T), g:((S→T)→ℕn). InvFuns(ℕn; S→T; f; g))
| by RWH (UnfoldC ‘inv_funs‘) 0
|
⊢  ∀S,T:Type.
|  (∃n:ℕ, f:(ℕn→S), g:(S→ℕn). g o f = Id & f o g = Id)
|  & (∃n:ℕ, f:(ℕn→T), g:(T→ℕn). g o f = Id & f o g = Id)
|  ⇒
|  (∃n:ℕ, f:(ℕn→S→T), g:((S→T)→ℕn). g o f = Id & f o g = Id)
| by GenUnivCD
|
1. S : Type
2. T : Type
3. (∃n:ℕ, f:(ℕn→S), g:(S→ℕn). g o f = Id & f o g = Id)
|  & (∃n:ℕ, f:(ℕn→T), g:(T→ℕn). g o f = Id & f o g = Id)
⊢  ∃n:ℕ, f:(ℕn→S→T), g:((S→T)→ℕn). g o f = Id & f o g = Id
| by Analyze 3 THEN Analyze 3
|
3. n : ℕ
4. ∃f:(ℕn→S), g:(S→ℕn). g o f = Id & f o g = Id
5. ∃n:ℕ, f:(ℕn→T), g:(T→ℕn). g o f = Id & f o g = Id
⊢   by Analyze 4 THEN Analyze 5
|
4. f : ℕn→S
5. g : S→ℕn
6. g o f = Id & f o g = Id
7. ∃n:ℕ, f:(ℕn→T), g:(T→ℕn). g o f = Id & f o g = Id
⊢   by Analyze 7 THEN Analyze 8 THEN Analyze 9
|
7. n1 : ℕ
8. f1 : ℕn1→T
9. g1 : T→ℕn1
10. g1 o f1 = Id & f1 o g1 = Id
⊢
| by Inst
|    Thm* ∀n,m:ℕ. ∃f:((ℕm→ℕn)→ℕ(n↑m)). Bij(ℕm→ℕn; ℕ(n↑m); f)
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|     {fun_enumer}
|     [n1;n]
|
11. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)). Bij(ℕn→ℕn1; ℕ(n1↑n); f)
⊢
| by RWH
|     (LemmaC
|      Thm* (∃f:(A→B). Bij(A; B; f)) ⟺ (A ∼ B)    {bij_iff_1_1_corr})
|     11
|
11. (ℕn→ℕn1) ∼ ℕ(n1↑n)
⊢   by Unfold ‘one_one_corr‘ 11
|
11. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)), g:(ℕ(n1↑n)→ℕn→ℕn1).
|   InvFuns(ℕn→ℕn1; ℕ(n1↑n); f; g)
⊢   by Unfold ‘inv_funs‘ 11
|
11. ∃f:((ℕn→ℕn1)→ℕ(n1↑n)), g:(ℕ(n1↑n)→ℕn→ℕn1).
|   g o f = Id & f o g = Id
⊢   by Analyze 11 THEN Analyze 12
|
11. f2 : (ℕn→ℕn1)→ℕ(n1↑n)
12. g2 : ℕ(n1↑n)→ℕn→ℕn1
13. g2 o f2 = Id & f2 o g2 = Id
⊢   by Analyze 10 THEN Analyze 14
|
10. g1 o f1 = Id
11. f1 o g1 = Id
12. f2 : (ℕn→ℕn1)→ℕ(n1↑n)
13. g2 : ℕ(n1↑n)→ℕn→ℕn1
14. g2 o f2 = Id
15. f2 o g2 = Id
⊢   by Witness (n1↑n)
|
⊢  ∃f:(ℕ(n1↑n)→S→T), g:((S→T)→ℕ(n1↑n)). g o f = Id & f o g = Id
| by Witness λh.f1 o g2(h) o g
|
⊢  ∃g@0:((S→T)→ℕ(n1↑n)).
|  g@0 o (λh.f1 o g2(h) o g) = Id & (λh.f1 o g2(h) o g) o g@0 = Id
| by Witness λh.f2(g1 o h o f)
|
⊢  (λh.f2(g1 o h o f)) o (λh.f1 o g2(h) o g) = Id
|  & (λh.f1 o g2(h) o g) o (λh.f2(g1 o h o f)) = Id  by Analyze 0
|
|\
| ⊢  (λh.f2(g1 o h o f)) o (λh.f1 o g2(h) o g) = Id  by Ext
| |
| 16. x : ℕ(n1↑n)
| ⊢  ((λh.f2(g1 o h o f)) o (λh.f1 o g2(h) o g))(x) = Id(x)
| | by RWH rem_composeC 0
| |
| ⊢  (λh.f2(g1 o h o f))((λh.f1 o g2(h) o g)(x)) = Id(x)  by Reduce 0
| |
| ⊢  f2(g1 o f1 o g2(x) o g o f) = x
| | by RWH
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| |    (RevLemmaC
| |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
| |          h o (g o f) = (h o g) o f ∈ A→D
| |     {comp_assoc})
| |    0
| |
| ⊢  f2(g1 o f1 o g2(x) o g o f) = x
| | by RWH
| |    (LemmaC
| |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
| |          h o (g o f) = (h o g) o f ∈ A→D
| |     {comp_assoc})
| |    0
| |
| ⊢  f2(g1 o f1 o g2(x) o g o f) = x  by RWH (HypC 10) 0
| |
| ⊢  f2(Id o g2(x) o g o f) = x
| | by RWH
| |    (RevLemmaC
| |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
| |          h o (g o f) = (h o g) o f ∈ A→D
| |     {comp_assoc})
| |    0
| |
| ⊢  f2(Id o g2(x) o g o f) = x  by Analyze 6
| |
| 6. g o f = Id
| 7. f o g = Id
| 8. n1 : ℕ
| 9. f1 : ℕn1→T
| 10. g1 : T→ℕn1
| 11. g1 o f1 = Id
| 12. f1 o g1 = Id
| 13. f2 : (ℕn→ℕn1)→ℕ(n1↑n)
| 14. g2 : ℕ(n1↑n)→ℕn→ℕn1
| 15. g2 o f2 = Id
| 16. f2 o g2 = Id
| 17. x : ℕ(n1↑n)
| ⊢   by RWH (HypC 6) 0
| |
| ⊢  f2(Id o g2(x) o Id) = x  by Unfold ‘compose‘ 0 THEN Reduce 0
| |
| ⊢  f2(λx@0.g2(x,x@0)) = x  by Assert ((λx@0.g2(x,x@0)) = g2(x))
| |
| |\
| | .....assertion .....
| | ⊢  (λx@0.g2(x,x@0)) = g2(x)  by Ext
| | |
| | 18. x1 : ℕn
| | ⊢  (λx@0.g2(x,x@0))(x1) = g2(x,x1)  by Reduce 0
| |
| .\
| . 18. (λx@0.g2(x,x@0)) = g2(x)
| . ⊢   by RWH (HypC 18) 0
| . |
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| . ⊢  f2(g2(x)) = x  by RWH add_composeC 0
| . |
| . ⊢  (f2 o g2)(x) = x  by RWH (HypC 16) 0
| . |
| . ⊢  Id(x) = x  by Reduce 0
|
.\
. ⊢  (λh.f1 o g2(h) o g) o (λh.f2(g1 o h o f)) = Id  by Ext
. |
. 16. x : S→T
. ⊢  ((λh.f1 o g2(h) o g) o (λh.f2(g1 o h o f)))(x) = Id(x)
. | by RWH rem_composeC 0
. |
. ⊢  (λh.f1 o g2(h) o g)((λh.f2(g1 o h o f))(x)) = Id(x)  by Reduce 0
. |
. ⊢  f1 o g2(f2(g1 o x o f)) o g = x
. | by RWH
. |    (LemmaC
. |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. |          h o (g o f) = (h o g) o f ∈ A→D
. |     {comp_assoc})
. |    0
. |
. ⊢  f1 o g2(f2(g1 o x o f)) o g = x  by RWH add_composeC 0
. |
. ⊢  f1 o (g2 o f2)(g1 o x o f) o g = x  by RWH (HypC 14) 0
. |
. ⊢  f1 o Id(g1 o x o f) o g = x  by Reduce 0
. |
. ⊢  f1 o g1 o x o f o g = x
. | by RWH
. |    (LemmaC
. |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. |          h o (g o f) = (h o g) o f ∈ A→D
. |     {comp_assoc})
. |    0
. |

. ⊢  f1 o g1 o x o f o g = x  by RWH (HypC 11) 0
. |
. ⊢  Id o x o f o g = x
. | by RWH
. |    (RevLemmaC
. |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. |          h o (g o f) = (h o g) o f ∈ A→D
. |     {comp_assoc})
. |    0
. |
. ⊢  Id o x o f o g = x
. | by RWH
. |    (RevLemmaC
. |     Thm* ∀f:(A→B), g:(B→C), h:(C→D).
. |          h o (g o f) = (h o g) o f ∈ A→D
. |     {comp_assoc})
. |    0
. |
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. ⊢  Id o x o f o g = x  by Analyze 6
. |
. 6. g o f = Id
. 7. f o g = Id
. 8. n1 : ℕ
. 9. f1 : ℕn1→T
. 10. g1 : T→ℕn1
. 11. g1 o f1 = Id
. 12. f1 o g1 = Id
. 13. f2 : (ℕn→ℕn1)→ℕ(n1↑n)
. 14. g2 : ℕ(n1↑n)→ℕn→ℕn1
. 15. g2 o f2 = Id
. 16. f2 o g2 = Id
. 17. x : S→T
. ⊢   by RWH (HypC 7) 0
. |
. ⊢  Id o x o Id = x  by Unfold ‘compose‘ 0
. |
. ⊢  (λx@0.Id((λx@0.x(Id(x@0)))(x@0))) = x  by Reduce 0
. |
. ⊢  (λx@0.x(x@0)) = x  by Ext
. |
. 18. x1 : S
. ⊢  (λx@0.x(x@0))(x1) = x(x1)  by Reduce 0
END
```