```
geom_series_limit  THM

⊢  ∀q:ℕ+, a:ℕ. ∃n:ℕ. Σ(q↑n⁻) ≤ a < Σ(q↑n+1⁻)
| by Analyze 0 THEN Analyze 0 THEN NatInd 2
|
|\
| .....basecase .....
| 1. q : ℕ+
| ⊢  ∃n:ℕ. Σ(q↑n⁻) ≤ 0 < Σ(q↑n+1⁻)
| | by Witness 0 THEN Analyze 0 THEN RecUnfold ‘geom_series‘ 0
| |    THEN
| |    Reduce 0
| |
| |\
| | ⊢  0≤0  by Auto
| |
| .\
| . ⊢  0<Σ(q↑0⁻)+(q↑0)
| .   by RecUnfold ‘geom_series‘ 0 THEN RecUnfold ‘exp‘ 0
| .      THEN
| .      Reduce 0
|
.\
. .....upcase .....
. 1. q : ℕ+
. 2. a : ℤ
. 3. 0<a
. 4. ∃n:ℕ. Σ(q↑n⁻) ≤ a-1 < Σ(q↑n+1⁻)
. ⊢  ∃n:ℕ. Σ(q↑n⁻) ≤ a < Σ(q↑n+1⁻)  by Analyze 4
. |
. 4. n : ℕ
. 5. Σ(q↑n⁻) ≤ a-1 < Σ(q↑n+1⁻)
. ⊢   by Decide (a<Σ(q↑n+1⁻))
. |
. |\
. | 6. a<Σ(q↑n+1⁻)
. | ⊢   by Witness n
. |
. .\
. . 6. ¬a<Σ(q↑n+1⁻)
. . ⊢   by Witness n+1
. . |
. . 5. Σ(q↑n⁻)≤a-1
. . 6. a-1<Σ(q↑n+1⁻)
. . 7. ¬a<Σ(q↑n+1⁻)
. . ⊢  a<Σ(q↑n+1+1⁻)
. . | by Inst
. . |    Thm* ∀q,n:ℕ, i:ℕ+. Σ(q↑n⁻)+(q↑n)≤Σ(q↑n+i⁻)
. . |    {geom_speed_lbound}
. . |    [q;n+1;1]
. . |
. . 8. Σ(q↑n+1⁻)+(q↑n+1)≤Σ(q↑n+1+1⁻)
. . ⊢   by Inst Thm* ∀m:ℕ+, n:ℕ. 0<(m↑n)    {exp_non_zero} [q;n+1]
END
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