phole_aux THM

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⊢  ∀n:{1...}, f:(ℕ(n+1)→ℕn). ∃i:ℕ(n+1), j:ℕi. f(i) = f(j)
| by (BackThru
|      Thm* ∀i:ℤ, E:({i...}→Prop).
|           E(i) ⇒ (∀k:{i+1...}. E(k-1) ⇒ E(k)) ⇒ (∀k:{i...}. E(k))
|      {int_upper_ind})
|     THEN
|     RepD
|
|\
| 1. f : ℕ(1+1)→ℕ1
| ⊢  ∃i:ℕ(1+1), j:ℕi. f(i) = f(j)  by InstConcl [1;0]
|
.\
. 1. n : {1+1...}
. 2. ∀f:(ℕ(n-1+1)→ℕ(n-1)). ∃i:ℕ(n-1+1), j:ℕi. f(i) = f(j)
. 3. f : ℕ(n+1)→ℕn
. ⊢  ∃i:ℕ(n+1), j:ℕi. f(i) = f(j)
. | by Assert
. |     (∀iii:ℕ(n+1).
. |      (∀ii:{(iii+1)..(n+1)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. |      (∃i:ℕ(iii+1), j:ℕi. f(i) = f(j)))
. |
. |\
. | .....assertion .....
. | ⊢  ∀iii:ℕ(n+1).
. | |  (∀ii:{(iii+1)..(n+1)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. | |  (∃i:ℕ(iii+1), j:ℕi. f(i) = f(j))
. | | by (BackThru
. | |      Thm* ∀i:ℤ, j:{i+1...}, E:({i..j⁻}→Prop).
. | |           E(i) ⇒
. | |           (∀k:{(i+1)..j⁻}. E(k-1) ⇒ E(k)) ⇒ (∀k:{i..j⁻}. E(k))
. | |      {int_seg_ind})
. | |     THEN
. | |     (All ArithSimp)
. | |
. | |\
. | | 1. n : {2...}
. | | 2. ∀f:(ℕn→ℕ(-1+n)). ∃i:ℕn, j:ℕi. f(i) = f(j)
. | | 3. f : ℕ(1+n)→ℕn
. | | ⊢  (∀ii:{1..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. | | |  (∃i:ℕ1, j:ℕi. f(i) = f(j))  by Analyze 0
. | | |
. | | 4. ∀ii:{1..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)
. | | ⊢  ∃i:ℕ1, j:ℕi. f(i) = f(j)
. | | | by Inst [λx.if f(x)=_2n-1→ f(n) else f(x) fi] 2
. | | |
. | | |\
. | | | .....wf .....
. | | | ⊢  (λx.if f(x)=_2n-1→ f(n) else f(x) fi) ∈ ℕn→ℕ(-1+n)
. | | | | by Analyze THEN SplitOnConclITE
. | | | |
. | | | |\
. | | | | .....truecase .....
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. | | | | 5. x : ℕn
. | | | | 6. f(x) = n-1
. | | | | ⊢  f(n) ∈ ℕ(-1+n)  by Inst [n;x] 4
. | | | |
. | | | .\
. | | | . .....falsecase .....
. | | | . 5. x : ℕn
. | | | . 6. ¬f(x) = n-1
. | | | . ⊢  f(x) ∈ ℕ(-1+n)  by Auto
. | | |
. | | .\
. | | . 5. ∃i:ℕn, j:ℕi.
. | | . |  (λx.if f(x)=₂n-1→ f(n) else f(x) fi)(i)
. | | . |  =
. | | . |  (λx.if f(x)=₂n-1→ f(n) else f(x) fi)(j)
. | | . ⊢   by Analyze 5 THEN Analyze 6 THEN Reduce 7
. | | . |
. | | . 5. i : ℕn
. | | . 6. j : ℕi
. | | . 7. if f(i)=₂n-1→ f(n) else f(i) fi
. | | . |  =
. | | . |  if f(j)=₂n-1→ f(n) else f(j) fi
. | | . ⊢   by MoveToConcl 7 THEN SplitOnConclITEs
. | | . |
. | | . |\
. | | . | .....truecase .....
. | | . | 7. f(i) = n-1
. | | . | 8. f(j) = n-1
. | | . | ⊢  f(n) = f(n) ⇒ (∃i:ℕ1, j:ℕi. f(i) = f(j))
. | | . |   by Inst [i;j] 4
. | | . |
. | | . |\
. | | . | .....falsecase .....
. | | . | 7. f(i) = n-1
. | | . | 8. ¬f(j) = n-1
. | | . | ⊢  f(n) = f(j) ⇒ (∃i:ℕ1, j:ℕi. f(i) = f(j))
. | | . | | by Inst [n;j] 4
. | | . | |
. | | . | 9. ¬f(n) = f(j)
. | | . | ⊢  f(n) ∈ ℕ(-1+n)  by Inst [n;i] 4
. | | . |
. | | . |\
. | | . | .....truecase .....
. | | . | 7. ¬f(i) = n-1
. | | . | 8. f(j) = n-1
. | | . | ⊢  f(i) = f(n) ⇒ (∃i:ℕ1, j:ℕi. f(i) = f(j))
. | | . |   by (Inst [n;i] 4) THEN (Inst [n;j] 4)

. | | . |
. | | . .\
. | | . . .....falsecase .....
. | | . . 7. ¬f(i) = n-1
. | | . . 8. ¬f(j) = n-1
. | | . . ⊢  f(i) = f(j) ⇒ (∃i:ℕ1, j:ℕi. f(i) = f(j))
. | | . .   by Inst [i;j] 4
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. | .\
. | . 1. n : {2...}
. | . 2. ∀f:(ℕn→ℕ(-1+n)). ∃i:ℕn, j:ℕi. f(i) = f(j)
. | . 3. f : ℕ(1+n)→ℕn
. | . ⊢  ∀iii:{1..(1+n)⁻}.
. | . |  ((∀ii:{iii..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. | . |   (∃i:ℕiii, j:ℕi. f(i) = f(j)))
. | . |  ⇒
. | . |
. | . |  (∀ii:{(1+iii)..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. | . |  (∃i:ℕ(1+iii), j:ℕi. f(i) = f(j))  by RepD
. | . |
. | . 4. iii : {1..(1+n)⁻}
. | . 5. (∀ii:{iii..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. | . |  (∃i:ℕiii, j:ℕi. f(i) = f(j))
. | . 6. ∀ii:{(1+iii)..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)
. | . ⊢  ∃i:ℕ(1+iii), j:ℕi. f(i) = f(j)
. | . | by (Let (fi = f(iii))) THEN (Decide (∃x:ℕiii. fi = f(x)))
. | . |
. | . |\
. | . | 7. fi : ℕn
. | . | 8. fi = f(iii)
. | . | 9. ∃x:ℕiii. fi = f(x)
. | . | ⊢   by Analyze -1 THEN Inst [iii;x] 0
. | . |
. | . .\
. | . . 7. fi : ℕn
. | . . 8. fi = f(iii)
. | . . 9. ¬(∃x:ℕiii. fi = f(x))
. | . . ⊢   by RWW "not_over_exists" -1
. | . . |
. | . . 9. ∀x:ℕiii. ¬fi = f(x)
. | . . ⊢   by (Analyze 5) THEN (Analyze -1) THEN (Analyze -1)
. | . . |
. | . . |\
. | . . | .....antecedent .....
. | . . | 5. ∀ii:{(1+iii)..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)
. | . . | 6. fi : ℕn
. | . . | 7. fi = f(iii)
. | . . | 8. ∀x:ℕiii. ¬fi = f(x)
. | . . | ⊢  ∀ii:{iii..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)
. | . . | | by (Analyze 0) THEN (Decide (iii = ii))
. | . . | |
. | . . | |\
. | . . | | 9. ii : {iii..(1+n)⁻}
. | . . | | 10. iii = ii
. | . . | | ⊢  ∀jj:ℕii. ¬f(ii) = f(jj)
. | . . | |   by (Analyze 0) THEN Witness8 jj THEN (Analyze 0)
. | . . | |      THEN
. | . . | |      (Analyze -2)
. | . . | |
. | . . | .\
. | . . | . 9. ii : {iii..(1+n)⁻}
. | . . | . 10. ¬iii = ii
. | . . | . ⊢  ∀jj:ℕii. ¬f(ii) = f(jj)
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. | . . | .   by (Analyze 0) THEN (BackThru 5)
. | . . |
. | . . .\
. | . . . 5. ∀ii:{(1+iii)..(1+n)⁻}, jj:ℕii. ¬f(ii) = f(jj)
. | . . . 6. fi : ℕn
. | . . . 7. fi = f(iii)
. | . . . 8. ∀x:ℕiii. ¬fi = f(x)
. | . . . 9. i : ℕiii
. | . . . 10. j : ℕi
. | . . . 11. f(i) = f(j)
. | . . . ⊢   by InstConcl [i;j]
. |
. .\
. . 4. ∀iii:ℕ(n+1).
. .    (∀ii:{(iii+1)..(n+1)⁻}, jj:ℕii. ¬f(ii) = f(jj)) ⇒
. .    (∃i:ℕ(iii+1), j:ℕi. f(i) = f(j))
. . ⊢   by Inst [n] 4
END
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