```
rem_add  THM

⊢  ∀x,y:ℤ, n:ℤ⁻⁰.
|  ((x+y) rem n)
|  =
|  (((x rem n)+(y rem n)) rem n)
|  +if
|    (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
|   ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
|   else 0 fi
| by ((Auto
|      THEN
|      (Inst
|       Thm* ∀a:ℤ, n:ℤ⁻⁰.
|             a = (a ÷ n)·n+(a rem n)
|             & |a rem n|<|n|
|             & ((a rem n)<0 ⇒ a<0)
|             & ((a rem n)>0 ⇒ a>0)
|       {div_rem_properties}
|       [x;n]))
|     THEN
|     (Inst
|      Thm* ∀a:ℤ, n:ℤ⁻⁰.
|            a = (a ÷ n)·n+(a rem n)
|            & |a rem n|<|n|
|            & ((a rem n)<0 ⇒ a<0)
|            & ((a rem n)>0 ⇒ a>0)
|      {div_rem_properties}
|      [y;n])
|     THEN
|     (Inst
|      Thm* ∀a:ℤ, n:ℤ⁻⁰.
|            a = (a ÷ n)·n+(a rem n)
|            & |a rem n|<|n|
|            & ((a rem n)<0 ⇒ a<0)
|            & ((a rem n)>0 ⇒ a>0)
|      {div_rem_properties}
|      [(x rem n)+(y rem n);n]))
|    THEN
|    RepDHyps
|    THEN
|    (Decide (x+y = 0))
|
|\
| 1. x : ℤ
| 2. y : ℤ
| 3. n : ℤ⁻⁰
| 4. x = (x ÷ n)·n+(x rem n)
| 5. |x rem n|<|n|
| 6. (x rem n)<0 ⇒ x<0
| 7. (x rem n)>0 ⇒ x>0
| 8. y = (y ÷ n)·n+(y rem n)
| 9. |y rem n|<|n|
| 10. (y rem n)<0 ⇒ y<0
| 11. (y rem n)>0 ⇒ y>0
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| 12. (x rem n)+(y rem n)
| |   =
| |   (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
| 13. |((x rem n)+(y rem n)) rem n|<|n|
| 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
| 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
| 16. x+y = 0
| ⊢  ((x+y) rem n)
| |  =
| |  (((x rem n)+(y rem n)) rem n)
| |  +if
| |    (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
| |   ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
| |   else 0 fi
| | by (HypSubst' -1 0) THEN (Subst' ((x rem n)+(y rem n) = 0) 0)
| |    THEN
| |    (Reduce 0)
| |
| |\
| | 1. x : ℤ
| | 2. y : ℤ
| | 3. n : ℤ^-0
| | 4. x = (x ÷ n)·n+(x rem n)
| | 5. |x rem n|<|n|
| | 6. (x rem n)<0 ⇒ x<0
| | 7. (x rem n)>0 ⇒ x>0
| | 8. y = (y ÷ n)·n+(y rem n)
| | 9. |y rem n|<|n|
| | 10. (y rem n)<0 ⇒ y<0
| | 11. (y rem n)>0 ⇒ y>0
| | 12. (x rem n)+(y rem n)
| |     =
| |     (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
| | 13. |((x rem n)+(y rem n)) rem n|<|n|
| | 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
| | 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
| | 16. x+y = 0
| | ⊢  (x rem n)+(y rem n) = 0
| |   by (Subst (y = -x) 0)
| |      THEN
| |      (RWO
| |       Thm* ∀x:ℤ, n:ℤ^-0. ((-x) rem n) = -(x rem n)    {rem_minus}
| |       0)
| |
| .\
| . 1. x : ℤ
| . 2. y : ℤ
| . 3. n : ℤ^-0
| . 4. x = (x ÷ n)·n+(x rem n)
| . 5. |x rem n|<|n|
| . 6. (x rem n)<0 ⇒ x<0
| . 7. (x rem n)>0 ⇒ x>0
| . 8. y = (y ÷ n)·n+(y rem n)
| . 9. |y rem n|<|n|
| . 10. (y rem n)<0 ⇒ y<0
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| . 11. (y rem n)>0 ⇒ y>0
| . 12. (x rem n)+(y rem n)
| . |   =
| . |   (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
| . 13. |((x rem n)+(y rem n)) rem n|<|n|
| . 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
| . 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
| . 16. x+y = 0
| . ⊢  (0 rem n) = (0 rem n)+if ((0 rem n)<₂0)∧₂false₂→ |n| else 0 fi
| . | by (Subst' ((0 rem n) = 0) 0) THEN (Reduce 0)
| . |
| . 1. x : ℤ
| . 2. y : ℤ
| . 3. n : ℤ^-0
| . 4. x = (x ÷ n)·n+(x rem n)
| . 5. |x rem n|<|n|
| . 6. (x rem n)<0 ⇒ x<0
| . 7. (x rem n)>0 ⇒ x>0
| . 8. y = (y ÷ n)·n+(y rem n)
| . 9. |y rem n|<|n|
| . 10. (y rem n)<0 ⇒ y<0
| . 11. (y rem n)>0 ⇒ y>0
| . 12. (x rem n)+(y rem n)
| .     =
| .     (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
| . 13. |((x rem n)+(y rem n)) rem n|<|n|
| . 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
| . 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
| . 16. x+y = 0
| . ⊢  (0 rem n) = 0
| .   by BackThru Thm* ∀n:ℤ^-0. (0 rem n) = 0    {zero_rem}
|
.\
. 1. x : ℤ
. 2. y : ℤ
. 3. n : ℤ^-0
. 4. x = (x ÷ n)·n+(x rem n)
. 5. |x rem n|<|n|
. 6. (x rem n)<0 ⇒ x<0
. 7. (x rem n)>0 ⇒ x>0
. 8. y = (y ÷ n)·n+(y rem n)
. 9. |y rem n|<|n|
. 10. (y rem n)<0 ⇒ y<0
. 11. (y rem n)>0 ⇒ y>0
. 12. (x rem n)+(y rem n)
. |   =
. |   (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. 13. |((x rem n)+(y rem n)) rem n|<|n|
. 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. 16. ¬x+y = 0
. ⊢  ((x+y) rem n)
. |  =
. |  (((x rem n)+(y rem n)) rem n)
. |  +if
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. |    (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
. |   ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
. |   else 0 fi
. | by Inst
. |    Thm* ∀a:ℤ, n:ℤ⁻⁰, q,r:ℤ.
. |         a = q·n+r
. |         ⇒
. |         |r|<|n|
. |         ⇒
. |         (r<0 ⇒ a<0)
. |         ⇒
. |         (r>0 ⇒ a>0) ⇒ q = (a ÷ n) & r = (a rem n)
. |    {div_rem_unique}
. |    [x+y
. |    ;n
. |    ;(x ÷ n)+(y ÷ n)+(((x rem n)+(y rem n)) ÷ n)
. |     +if
. |       (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→...
. |      ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→...
. |      else 0 fi
. |    ;(((x rem n)+(y rem n)) rem n)
. |     +if
. |       (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
. |      ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
. |      else 0 fi]
. |
. |\
. | .....antecedent .....
. | 1. x : ℤ
. | 2. y : ℤ
. | 3. n : ℤ⁻⁰
. | 4. x = (x ÷ n)·n+(x rem n)
. | 5. |x rem n|<|n|
. | 6. (x rem n)<0 ⇒ x<0
. | 7. (x rem n)>0 ⇒ x>0
. | 8. y = (y ÷ n)·n+(y rem n)
. | 9. |y rem n|<|n|
. | 10. (y rem n)<0 ⇒ y<0
. | 11. (y rem n)>0 ⇒ y>0
. | 12. (x rem n)+(y rem n)
. |      =
. |      (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. | 13. |((x rem n)+(y rem n)) rem n|<|n|
. | 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. | 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. | 16. ¬x+y = 0
. | ⊢  x+y
. |    =
. |    ((x ÷ n)+(y ÷ n)+(((x rem n)+(y rem n)) ÷ n)
. |     +if
. |       (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→...
. |      ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→...
. |      else 0 fi)·n+(((x rem n)+(y rem n)) rem n)
. |    +if
. |      (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
```

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. |      ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
. |      else 0 fi
. |    by (Unfold `absval` 0) THEN (AutoBoolCase (x+y<₂0))
. |       THEN
. |       SplitOnConclITE
. |       THEN
. |       (AutoBoolCase (0<₂x+y))
. |       THEN
. |       SplitOnConclITE
. |       THEN
. |       SplitOnConclITE
. |
. |\
. | .....antecedent .....
. | 1. x : ℤ
. | 2. y : ℤ
. | 3. n : ℤ^-0
. | 4. x = (x ÷ n)·n+(x rem n)
. | 5. |x rem n|<|n|
. | 6. (x rem n)<0 ⇒ x<0
. | 7. (x rem n)>0 ⇒ x>0
. | 8. y = (y ÷ n)·n+(y rem n)
. | 9. |y rem n|<|n|
. | 10. (y rem n)<0 ⇒ y<0
. | 11. (y rem n)>0 ⇒ y>0
. | 12. (x rem n)+(y rem n)
. |      =
. |      (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. | 13. |((x rem n)+(y rem n)) rem n|<|n|
. | 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. | 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. | 16. ¬x+y = 0
. | ⊢  |(((x rem n)+(y rem n)) rem n)
. |     +if
. |       (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
. |      ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
. |      else 0 fi|<|n|
. |    by (MoveToConcl -4) THEN (Unfold `absval` 0)
. |       THEN
. |       (RepeatFor 2 SplitOnConclITE)
. |       THEN
. |       (AutoBoolCase (x+y<₂0))
. |       THEN
. |       (AutoBoolCase (0<₂x+y))
. |       THEN
. |       (AutoBoolCase ((((x rem n)+(y rem n)) rem n)<₂0))
. |       THEN
. |       (AutoBoolCase (0<₂(((x rem n)+(y rem n)) rem n)))
. |       THEN
. |       SplitOnConclITE
. |
. |\
. | 1. x : ℤ
. | 2. y : ℤ
. | 3. n : ℤ^-0
```

```
. | 4. x = (x ÷ n)·n+(x rem n)
. | 5. |x rem n|<|n|
. | 6. (x rem n)<0 ⇒ x<0
. | 7. (x rem n)>0 ⇒ x>0
. | 8. y = (y ÷ n)·n+(y rem n)
. | 9. |y rem n|<|n|
. | 10. (y rem n)<0 ⇒ y<0
. | 11. (y rem n)>0 ⇒ y>0
. | 12. (x rem n)+(y rem n)
. | |   =
. | |   (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. | 13. |((x rem n)+(y rem n)) rem n|<|n|
. | 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. | 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. | 16. ¬x+y = 0
. | 17. (((x rem n)+(y rem n)) rem n)
. | |   +if
. | |     (x+y<₂0)∧₂(0<₂(((x rem n)+(y rem n)) rem n))→-|n|
. | |    ;((((x rem n)+(y rem n)) rem n)<₂0)∧₂(0<₂x+y)→|n|
. | |    else 0 fi<0
. | ⊢  x+y<0
. | | by (MoveToConcl -1) THEN (AutoBoolCase (x+y<₂0))
. | |    THEN
. | |    (AutoBoolCase ((((x rem n)+(y rem n)) rem n)<₂0))
. | |    THEN
. | |    ThinTrivial
. | |    THEN
. | |    (AutoBoolCase (0<₂x+y))
. | |
. | 1. x : ℤ
. | 2. y : ℤ
. | 3. n : ℤ⁻⁰
. | 4. x = (x ÷ n)·n+(x rem n)
. | 5. |x rem n|<|n|
. | 6. (x rem n)<0 ⇒ x<0
. | 7. (x rem n)>0 ⇒ x>0
. | 8. y = (y ÷ n)·n+(y rem n)
. | 9. |y rem n|<|n|
. | 10. (y rem n)<0 ⇒ y<0
. | 11. (y rem n)>0 ⇒ y>0
. | 12. (x rem n)+(y rem n)
. |     =
. |     (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. | 13. |((x rem n)+(y rem n)) rem n|<|n|
. | 14. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. | 15. ¬x+y = 0
. | 16. 0≤x+y
. | 17. (((x rem n)+(y rem n)) rem n)<0
. | 18. (x rem n)+(y rem n)<0
. | 19. 0<x+y
. | ⊢  (((x rem n)+(y rem n)) rem n)+|n|<0 ⇒ x+y<0
. |   by (MoveToConcl -7) THEN (Unfold ‘absval‘ 0)
. |      THEN
. |      (RepeatFor 2 SplitOnConclITE)
. |
```

```
. .\
. . 1. x : ℤ
. . 2. y : ℤ
. . 3. n : ℤ^-0
. . 4. x = (x ÷ n)·n+(x rem n)
. . 5. |x rem n|<|n|
. . 6. (x rem n)<0 ⇒ x<0
. . 7. (x rem n)>0 ⇒ x>0
. . 8. y = (y ÷ n)·n+(y rem n)
. . 9. |y rem n|<|n|
. . 10. (y rem n)<0 ⇒ y<0
. . 11. (y rem n)>0 ⇒ y>0
. . 12. (x rem n)+(y rem n)
. . |   =
. . |   (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. . 13. |((x rem n)+(y rem n)) rem n|<|n|
. . 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. . 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. . 16. ¬x+y = 0
. . 17. (((x rem n)+(y rem n)) rem n)
. . |   +if
. . |     (x+y<_2 0)∧_2(0<_2(((x rem n)+(y rem n)) rem n))→-|n|
. . |    ;((((x rem n)+(y rem n)) rem n)<_2 0)∧_2(0<_2 x+y)→|n|
. . |    else 0 fi>0
. . ⊢  x+y>0
. . | by (MoveToConcl -1) THEN (AutoBoolCase (x+y<_2 0))
. . |    THEN
. . |    (AutoBoolCase (0<_2(((x rem n)+(y rem n)) rem n)))
. . |    THEN
. . |    (AutoBoolCase ((((x rem n)+(y rem n)) rem n)<_2 0))
. . |    THEN
. . |    ThinTrivial
. . |    THEN
. . |    (AutoBoolCase (0<_2 x+y))
. . |
. . 1. x : ℤ
. . 2. y : ℤ
. . 3. n : ℤ^-0
. . 4. x = (x ÷ n)·n+(x rem n)
. . 5. |x rem n|<|n|
. . 6. (x rem n)<0 ⇒ x<0
. . 7. (x rem n)>0 ⇒ x>0
. . 8. y = (y ÷ n)·n+(y rem n)
. . 9. |y rem n|<|n|
. . 10. (y rem n)<0 ⇒ y<0
. . 11. (y rem n)>0 ⇒ y>0
. . 12. (x rem n)+(y rem n)
. .     =
. .     (((x rem n)+(y rem n)) ÷ n)·n+(((x rem n)+(y rem n)) rem n)
. . 13. |((x rem n)+(y rem n)) rem n|<|n|
. . 14. (((x rem n)+(y rem n)) rem n)<0 ⇒ (x rem n)+(y rem n)<0
. . 15. (((x rem n)+(y rem n)) rem n)>0 ⇒ (x rem n)+(y rem n)>0
. . 16. ¬x+y = 0
. . 17. x+y<0
. . 18. 0<(((x rem n)+(y rem n)) rem n)
```

```
. . 19. 0≤(((x rem n)+(y rem n)) rem n)
. . 20. x+y≤0
. . ⊢  (((x rem n)+(y rem n)) rem n)+-|n|>0 ⇒ x+y>0
. .   by (MoveToConcl -8) THEN (Unfold `absval` 0)
. .      THEN
. .      (RepeatFor 2 SplitOnConclITE)
END
```