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topsort-properties  THM

⊢  For any graph
|  ∀the_obj:GraphObject(the_graph).
|  (non-trivial-loop-free(the_graph)
|   ⇒
|   topsorted(the_graph;topsort(the_obj)))
|  & (∀i:V. (i ∈ topsort(the_obj)))
|  & no_repeats(V;topsort(the_obj))
| by (RepeatFor 2 (Analyze 0))
|    THEN
|    (Inst
|     Thm* For any graph
|          ∀the_obj:GraphObject(the_graph).
|          paren(V;depthfirst(the_obj))
|          & no_repeats(V+V;depthfirst(the_obj))
|          & df-traversal(the_graph;depthfirst(the_obj))
|          & depthfirst-traversal(the_graph;depthfirst(the_obj))
|          & (∀i:V.
|             (inr(i) ∈ depthfirst(the_obj))
|             & (inl(i) ∈ depthfirst(the_obj)))
|     {depthfirst_DASH_properties}
|     [the_graph;the_obj])
|
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| 1. the_graph : Graph
| 2. the_obj : GraphObject(the_graph)
| 3. paren(Vertices(the_graph);depthfirst(the_obj))
| 4. no_repeats(Vertices(the_graph)+
| |  Vertices(the_graph);depthfirst(the_obj))
| 5. df-traversal(the_graph;depthfirst(the_obj))
| 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
| 7. ∀i:Vertices(the_graph).
| |  (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
| 8. non-trivial-loop-free(the_graph)
| ⊢  topsorted(the_graph;topsort(the_obj))  by Unfold ‘topsorted‘ 0
| |
| 1. the_graph : Graph
| 2. the_obj : GraphObject(the_graph)
| 3. paren(Vertices(the_graph);depthfirst(the_obj))
| 4. no_repeats(Vertices(the_graph)+
| |  Vertices(the_graph);depthfirst(the_obj))
| 5. df-traversal(the_graph;depthfirst(the_obj))
| 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
| 7. ∀i:Vertices(the_graph).
| |  (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
| 8. non-trivial-loop-free(the_graph)
| 9. i : Vertices(the_graph)
| 10. j : Vertices(the_graph)
| 11. i-the_graph->*j
| 12. ¬i = j
| ⊢  i before j ∈ topsort(the_obj)
| | by Assert inl(i) before inl(j) ∈ depthfirst(the_obj)
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| | .....assertion .....
| | 1. the_graph : Graph
| | 2. the_obj : GraphObject(the_graph)
| | 3. paren(Vertices(the_graph);depthfirst(the_obj))
| | 4. no_repeats(Vertices(the_graph)+
| | |  Vertices(the_graph);depthfirst(the_obj))
| | 5. df-traversal(the_graph;depthfirst(the_obj))
| | 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
| | 7. ∀i:Vertices(the_graph).
| | |  (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
| | 8. non-trivial-loop-free(the_graph)
| | 9. i : Vertices(the_graph)
| | 10. j : Vertices(the_graph)
| | 11. i-the_graph->*j
| | 12. ¬i = j
| | ⊢  inl(i) before inl(j) ∈ depthfirst(the_obj)
| | | by (InstHyp [i] -6) THEN (ListMemD -1)
| | |    THEN
| | |    (AllHyps
| | |     (λh.
| | |      ((Unfold `depthfirst-traversal` h)
| | |       THEN
| | |       (InstHyp [i;s1;s2;j] h))
| | |      THENA
| | |      (Reduce 0)))
| | |
| | |\
| | | 1. the_graph : Graph
| | | 2. the_obj : GraphObject(the_graph)
| | | 3. paren(Vertices(the_graph);depthfirst(the_obj))
| | | 4. no_repeats(Vertices(the_graph)+
| | |    Vertices(the_graph);depthfirst(the_obj))
| | | 5. df-traversal(the_graph;depthfirst(the_obj))
| | | 6. ∀i:Vertices(the_graph), s1,s2:traversal(the_graph).
| | |    (∀j:Vertices(the_graph).
| | |     i-the_graph->*j ⇒ ¬non-trivial-loop(the_graph;j))
| | |    ⇒
| | |    depthfirst(the_obj) = (s1 @ [inl(i)] @ s2)
| | |    ⇒
| | |    (∀j:Vertices(the_graph).
| | |     ¬j = i ⇒ i-the_graph->*j ⇒ (inl(j) ∈ s2))
| | | 7. ∀i:Vertices(the_graph).
| | |    (inr(i) ∈ depthfirst(the_obj))
| | |    & (inl(i) ∈ depthfirst(the_obj))
| | | 8. non-trivial-loop-free(the_graph)
| | | 9. i : Vertices(the_graph)
| | | 10. j : Vertices(the_graph)
| | | 11. i-the_graph->*j
| | | 12. ¬i = j
| | | 13. (inr(i) ∈ depthfirst(the_obj))
| | | 14. (inl(i) ∈ depthfirst(the_obj))
| | | 15. s1 : (Vertices(the_graph)+Vertices(the_graph)) List
| | | 16. s2 : (Vertices(the_graph)+Vertices(the_graph)) List
| | | 17. depthfirst(the_obj) = (s1 @ [inl(i) / s2])
| | | 18. j1 : Vertices(the_graph)
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| | | 19. i-the_graph->*j1
| | | ⊢  ¬non-trivial-loop(the_graph;j1)
| | |   by AllHyps
| | |      (λh.(Unfold ‘non-trivial-loop-free‘ h) THEN (BackThru h))
| | |
| | |\
| | | .....antecedent .....
| | | 1. the_graph : Graph
| | | 2. the_obj : GraphObject(the_graph)
| | | 3. paren(Vertices(the_graph);depthfirst(the_obj))
| | | 4. no_repeats(Vertices(the_graph)+
| | |    Vertices(the_graph);depthfirst(the_obj))
| | | 5. df-traversal(the_graph;depthfirst(the_obj))
| | | 6. ∀i:Vertices(the_graph), s1,s2:traversal(the_graph).
| | |    (∀j:Vertices(the_graph).
| | |     i-the_graph->*j ⇒ ¬non-trivial-loop(the_graph;j))
| | |    ⇒
| | |    depthfirst(the_obj) = (s1 @ [inl(i)] @ s2)
| | |    ⇒
| | |    (∀j:Vertices(the_graph).
| | |     ¬j = i ⇒ i-the_graph->*j ⇒ (inl(j) ∈ s2))
| | | 7. ∀i:Vertices(the_graph).
| | |    (inr(i) ∈ depthfirst(the_obj))
| | |    & (inl(i) ∈ depthfirst(the_obj))
| | | 8. non-trivial-loop-free(the_graph)
| | | 9. i : Vertices(the_graph)
| | | 10. j : Vertices(the_graph)
| | | 11. i-the_graph->*j
| | | 12. ¬i = j
| | | 13. (inr(i) ∈ depthfirst(the_obj))
| | | 14. (inl(i) ∈ depthfirst(the_obj))
| | | 15. s1 : (Vertices(the_graph)+Vertices(the_graph)) List
| | | 16. s2 : (Vertices(the_graph)+Vertices(the_graph)) List
| | | 17. depthfirst(the_obj) = (s1 @ [inl(i) / s2])
| | | ⊢  ¬j = i  by AllHyps ParallelOp
| | |
| | .\
| | . 1. the_graph : Graph
| | . 2. the_obj : GraphObject(the_graph)
| | . 3. paren(Vertices(the_graph);depthfirst(the_obj))
| | . 4. no_repeats(Vertices(the_graph)+
| | . |  Vertices(the_graph);depthfirst(the_obj))
| | . 5. df-traversal(the_graph;depthfirst(the_obj))
| | . 6. ∀i:Vertices(the_graph), s1,s2:traversal(the_graph).
| | . |  (∀j:Vertices(the_graph).
| | . |   i-the_graph->*j ⇒ ¬non-trivial-loop(the_graph;j))
| | . |  ⇒
| | . |  depthfirst(the_obj) = (s1 @ [inl(i)] @ s2)
| | . |  ⇒
| | . |  (∀j:Vertices(the_graph).
| | . |   ¬j = i ⇒ i-the_graph->*j ⇒ (inl(j) ∈ s2))
| | . 7. ∀i:Vertices(the_graph).
| | . |  (inr(i) ∈ depthfirst(the_obj))
| | . |  & (inl(i) ∈ depthfirst(the_obj))
| | . 8. non-trivial-loop-free(the_graph)
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| | . 9. i : Vertices(the_graph)
| | . 10. j : Vertices(the_graph)
| | . 11. i-the_graph->*j
| | . 12. ¬i = j
| | . 13. (inr(i) ∈ depthfirst(the_obj))
| | . 14. (inl(i) ∈ depthfirst(the_obj))
| | . 15. s1 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 16. s2 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 17. depthfirst(the_obj) = (s1 @ [inl(i) / s2])
| | . 18. (inl(j) ∈ s2)
| | . ⊢  inl(i) before inl(j) ∈ depthfirst(the_obj)
| | . | by ((HypSubst -2 0)
| | . |     THEN
| | . |     (RWO
| | . |      Thm* ∀L:T List, x,y:T.
| | . |           x before y ∈ L
| | . |           ⟺
| | . |           (∃L1,L2,L3:T List. L = (L1 @ [x] @ L2 @ [y] @ L3))
| | . |      {l_before_DASH_iff}
| | . |      0))
| | . |    THEN
| | . |    (ListMemD -1)
| | . |    THEN
| | . |    (HypSubst -1 0)
| | . |    THEN
| | . |    (Reduce 0)
| | . |
| | . 1. the_graph : Graph
| | . 2. the_obj : GraphObject(the_graph)
| | . 3. paren(Vertices(the_graph);depthfirst(the_obj))
| | . 4. no_repeats(Vertices(the_graph)+
| | .    Vertices(the_graph);depthfirst(the_obj))
| | . 5. df-traversal(the_graph;depthfirst(the_obj))
| | . 6. ∀i:Vertices(the_graph), s1,s2:traversal(the_graph).
| | .    (∀j:Vertices(the_graph).
| | .     i-the_graph->*j ⇒ ¬non-trivial-loop(the_graph;j))
| | .    ⇒
| | .    depthfirst(the_obj) = (s1 @ [inl(i)] @ s2)
| | .    ⇒
| | .    (∀j:Vertices(the_graph).
| | .     ¬j = i ⇒ i-the_graph->*j ⇒ (inl(j) ∈ s2))
| | . 7. ∀i:Vertices(the_graph).
| | .    (inr(i) ∈ depthfirst(the_obj))
| | .    & (inl(i) ∈ depthfirst(the_obj))
| | . 8. non-trivial-loop-free(the_graph)
| | . 9. i : Vertices(the_graph)
| | . 10. j : Vertices(the_graph)
| | . 11. i-the_graph->*j
| | . 12. ¬i = j
| | . 13. (inr(i) ∈ depthfirst(the_obj))
| | . 14. (inl(i) ∈ depthfirst(the_obj))
| | . 15. s1 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 16. s2 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 17. depthfirst(the_obj) = (s1 @ [inl(i) / s2])
| | . 18. (inl(j) ∈ s2)
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| | . 19. s3 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 20. s2@0 : (Vertices(the_graph)+Vertices(the_graph)) List
| | . 21. s2 = (s3 @ [inl(j) / s2@0])
| | . ⊢  ∃L1,L2,L3:(Vertices(the_graph)+Vertices(the_graph)) List.
| | .    (s1 @ [inl(i) / (s3 @ [inl(j) / s2@0])])
| | .    =
| | .    (L1 @ [inl(i) / (L2 @ [inl(j) / L3])])
| | .   by InstConcl [s1;s3;s2@0]
| |
| .\
| . 1. the_graph : Graph
| . 2. the_obj : GraphObject(the_graph)
| . 3. paren(Vertices(the_graph);depthfirst(the_obj))
| . 4. no_repeats(Vertices(the_graph)+
| .    Vertices(the_graph);depthfirst(the_obj))
| . 5. df-traversal(the_graph;depthfirst(the_obj))
| . 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
| . 7. ∀i:Vertices(the_graph).
| .    (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
| . 8. non-trivial-loop-free(the_graph)
| . 9. i : Vertices(the_graph)
| . 10. j : Vertices(the_graph)
| . 11. i-the_graph->*j
| . 12. ¬i = j
| . 13. inl(i) before inl(j) ∈ depthfirst(the_obj)
| . ⊢  i before j ∈ topsort(the_obj)
| .   by (((Unfold `topsort` 0) THEN (Unfold `mapoutl` 0)
| .        THEN
| .        (Inst
| .         Thm* ∀P:(T→𝔹), T':Type, f:({x:T| P(x) }→T'), L:T List
| .              , x,y:{x:T| P(x) }.
| .              x before y ∈ L ⇒ f(x) before f(y) ∈ mapfilter(f;P;L)
| .         {mapfilter_before}
| .         [Vertices(the_graph)+Vertices(the_graph)
| .         ;λx.isl(x)
| .         ;Vertices(the_graph)
| .         ;λx.outl(x)
| .         ;depthfirst(the_obj)
| .         ;inl(i)
| .         ;inl(j)]))
| .       THENA
| .       (Auto THEN (All Reduce)
| .        THEN
| .        (Try ((Analyze -1) THEN Unhide THEN (Complete Auto)))
| .        THEN
| .        (Try (Analyze THEN (Reduce 0)))))
| .      THEN
| .      (Reduce -1)
| .      THEN
| .      (NthHyp -1)
|
|\
| 1. the_graph : Graph
| 2. the_obj : GraphObject(the_graph)
| 3. paren(Vertices(the_graph);depthfirst(the_obj))
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| 4. no_repeats(Vertices(the_graph)+
|    Vertices(the_graph);depthfirst(the_obj))
| 5. df-traversal(the_graph;depthfirst(the_obj))
| 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
| 7. ∀i:Vertices(the_graph).
|    (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
| 8. i : Vertices(the_graph)
| ⊢  (i ∈ topsort(the_obj))
|   by (Unfold `topsort` 0)
|      THEN
|      (RWO
|       Thm* ∀s:(A+B) List, x:A.
|            (x ∈ mapoutl(s))
|            ⟺
|            (∃y:A+B. (y ∈ s) & isl(y) & x = outl(y))
|       {member_mapoutl}
|       0)
|      THEN
|      (InstConcl [inl(i)])
|      THEN
|      (Reduce 0)
|      THEN
|      (InstHyp [i] -2)
|
.\
. 1. the_graph : Graph
. 2. the_obj : GraphObject(the_graph)

. 3. paren(Vertices(the_graph);depthfirst(the_obj))
. 4. no_repeats(Vertices(the_graph)+
.    Vertices(the_graph);depthfirst(the_obj))
. 5. df-traversal(the_graph;depthfirst(the_obj))
. 6. depthfirst-traversal(the_graph;depthfirst(the_obj))
. 7. ∀i:Vertices(the_graph).
.    (inr(i) ∈ depthfirst(the_obj)) & (inl(i) ∈ depthfirst(the_obj))
. ⊢  no_repeats(Vertices(the_graph);topsort(the_obj))
.   by (Unfold `topsort` 0)
.      THEN
.      (BackThru
.       Thm* ∀s:(A+B) List.
.            no_repeats(A+B;s) ⇒ no_repeats(A;mapoutl(s))
.       {no_repeats_mapoutl})
END
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