```
iter_via_intseg_comp_binop  THM

⊢  ∀A:Type, f:(A→A→A), u:A.
|  is_commutative_sep(A; f)
|  ⇒
|  is_ident(A; f; u)
|  ⇒
|  is_assoc_sep(A; f)
|  ⇒
|  (∀a,b:ℤ, e,g:({a..b⁻}→A).
|   f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
|   =
|   (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
|   ∈ A)  by Guarding (∀a:<type>. <prop>) Auto
|
1. A : Type
2. f : A→A→A
3. u : A
4. is_commutative_sep(A; f)
5. is_ident(A; f; u)
6. is_assoc_sep(A; f)
⊢  ∀a,b:ℤ, e,g:({a..b⁻}→A).
|  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
|  =
|  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
| by Guarding (∀a:<type>. <prop>)
|    BackThru:
|    Thm*  ∀P:(ℤ→ℤ→Prop).
|          (∀a:ℤ, b:{...a}. P(a,b))
|          ⇒
|          (∀a:ℤ. (∀b:{...a}. P(a,b)) ⇒ (∀b:{a...}. P(a,b)))
|          ⇒
|          (∀a,b:ℤ. P(a,b))
|    {all_int_pairs_via_all_splits}
|    THEN
|    At Type (Guarding (∀b:<type>. <prop>) Auto)
|
|\
| 7. a : ℤ
| ⊢  ∀b:{...a}, e,g:({a..b⁻}→A).
| |  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
| |  =
| |  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
| | by Rewrite by
| |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
| |          b≤a ⇒ (Iter(f;u) i:{a..b⁻}. e(i)) = u
| |    {iter_via_intseg_null}
| |
| 8. b : {...a}
| 9. {a..b⁻}→A
| 10. {a..b⁻}→A
| ⊢  f(u,u) = u  by BackThru: is_ident(A; f; u)
|
.\
. 7. a : ℤ
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. 8. ∀b:{...a}, e,g:({a..b⁻}→A).
. |  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. |  =
. |  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
. ⊢  ∀b:{a...}, e,g:({a..b⁻}→A).
. |  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. |  =
. |  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
. | by Inst:
. |    Thm*  ∀P:(ℤ→ℤ→Prop), a:ℤ.
. |          (∀b:{a...}. (∀c:{a..b⁻}. P(a,c)) ⇒ P(a,b))
. |          ⇒
. |          (∀b:{a...}. P(a,b))
. |    {int_seg_upper_ind}
. |    Using:[P(a,b):=
. |           ∀e,g:({a..b⁻}→A).
. |           f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. |           =
. |           (Iter(f;u) i:{a..b⁻}. f(e(i),g(i))) |
. |           a]
. |
. 9. b : {a...}
. 10. ∀c:{a..b⁻}, e,g:({a..c⁻}→A).
. |   f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{a..c⁻}. g(i))
. |   =
. |   (Iter(f;u) i:{a..c⁻}. f(e(i),g(i)))
. 11. e : {a..b⁻}→A
. 12. g : {a..b⁻}→A
. ⊢  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. |  =
. |  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))  by Decide: a = b ∈ ℤ
. |
. |\
. | 13. a = b
. | ⊢  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. |    =
. |    (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))  by BackThru: Hyp:8
. |
. .\
. . 13. ¬a = b
. . ⊢  f((Iter(f;u) i:{a..b⁻}. e(i)),Iter(f;u) i:{a..b⁻}. g(i))
. . |  =
. . |  (Iter(f;u) i:{a..b⁻}. f(e(i),g(i)))
. . | by Rewrite by
. . |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
. . |          a<b
. . |          ⇒

. . |          (Iter(f;u) i:{a..b⁻}. e(i))
. . |          =
. . |          f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. . |    {iter_via_intseg_split_last}
. . |
. . ⊢  f
. . |  (f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. . |  ,f((Iter(f;u) i:{a..b-1⁻}. g(i)),g(b-1)))
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. . |  =
. . |  f((Iter(f;u) i:{a..b-1⁻}. f(e(i),g(i))),f(e(b-1),g(b-1)))
. . | by Rewrite by Hyp:10
. . |
. . ⊢  f
. . |  (f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. . |  ,f((Iter(f;u) i:{a..b-1⁻}. g(i)),g(b-1)))
. . |  =
. . |  f
. . |  (f((Iter(f;u) i:{a..b-1⁻}. e(i)),Iter(f;u) i:{a..b-1⁻}. g(i))
. . |  ,f(e(b-1),g(b-1)))
. . | by BackThru:
. . |    ∀U,X,V,Y:A. f(f(U,X),f(V,Y)) = f(f(U,V),f(X,Y)) ∈ A
. . |
. . 14. U : A
. . 15. X : A
. . 16. V : A
. . 17. Y : A
. . ⊢  f(f(U,X),f(V,Y)) = f(f(U,V),f(X,Y))
. . | by Rewrite by is_assoc_sep(A; f)
. . |
. . ⊢  f(f(f(U,X),V),Y) = f(f(f(U,V),X),Y)  by 2 Times Analyze
. . |
. . ⊢  f(f(U,X),V) = f(f(U,V),X)  by Rewrite by is_assoc_sep(A; f)
. . |
. . ⊢  f(U,f(X,V)) = f(U,f(V,X))  by Analyze
. . |
. . ⊢  f(X,V) = f(V,X)  by BackThru: is_commutative_sep(A; f)
END
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