```
iter_via_intseg_split_mid  THM

⊢  ∀A:Type, f:(A→A→A), u:A.
|  is_ident(A; f; u)
|  ⇒
|  is_assoc_sep(A; f)
|  ⇒
|  (∀a,c,b:ℤ, e:({a..b⁻}→A).
|   a≤c
|   ⇒
|   c≤b
|   ⇒
|   (Iter(f;u) i:{a..b⁻}. e(i))
|   =
|   f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i)))
| by Guarding (∀b:<type>. <prop>) Auto
|
1. A : Type
2. f : A→A→A
3. u : A
4. is_ident(A; f; u)
5. is_assoc_sep(A; f)
6. a : ℤ
7. c : ℤ
⊢  ∀b:ℤ, e:({a..b⁻}→A).
|  a≤c
|  ⇒
|  c≤b
|  ⇒
|  (Iter(f;u) i:{a..b⁻}. e(i))
|  =
|  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
| by Decide: a = c ∈ ℤ
|
|\
| 8. a = c
| ⊢  ∀b:ℤ, e:({a..b⁻}→A).
| |  a≤c
| |  ⇒
| |  c≤b
| |  ⇒
| |  (Iter(f;u) i:{a..b⁻}. e(i))
| |  =
| |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
| | by Auto THEN Rewrite by a = c
| |
| 9. b : ℤ
| 10. e : {a..b⁻}→A
| 11. a≤c
| 12. c≤b
| ⊢  (Iter(f;u) i:{c..b⁻}. e(i))
| |  =
| |  f((Iter(f;u) i:{c..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
| | by Rewrite by
| |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
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| |          b≤a ⇒ (Iter(f;u) i:{a..b⁻}. e(i)) = u
| |    {iter_via_intseg_null}
| |    Using:[a:= c | b:= c]
| |
| ⊢  (Iter(f;u) i:{c..b⁻}. e(i)) = f(u,Iter(f;u) i:{c..b⁻}. e(i))
|   by Inst: is_ident(A; f; u) Using:[Iter(f;u) i:{c..b⁻}. e(i)]
|
.\
. 8. ¬a = c
. ⊢  ∀b:ℤ, e:({a..b⁻}→A).
. |  a≤c
. |  ⇒
. |  c≤b
. |  ⇒
. |  (Iter(f;u) i:{a..b⁻}. e(i))
. |  =
. |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. | by a<c
. |    ⇒
. |    (∀k:ℕ, b:ℤ.
. |     b-c = k
. |     ⇒
. |     (∀e:({a..b⁻}→A).
. |      (Iter(f;u) i:{a..b⁻}. e(i))
. |      =
. |      f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))))
. |    Asserted
. |
. |\
. | .....assertion .....
. | ⊢  a<c
. | |  ⇒
. | |  (∀k:ℕ, b:ℤ.
. | |   b-c = k
. | |   ⇒
. | |   (∀e:({a..b⁻}→A).
. | |    (Iter(f;u) i:{a..b⁻}. e(i))
. | |    =
. | |    f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))))
. | | by Analyze THEN CompNatInd Concl
. | |
. | 9. a<c
. | 10. k : ℕ
. | 11. ∀k1:ℕ.
. | |   k1<k
. | |   ⇒
. | |   (∀b:ℤ.

. | |    b-c = k1
. | |    ⇒
. | |    (∀e:({a..b⁻}→A).
. | |     (Iter(f;u) i:{a..b⁻}. e(i))
. | |     =
. | |     f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))))
. | 12. b : ℤ
. | 13. b-c = k
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. | 14. e : {a..b⁻}→A
. | ⊢  (Iter(f;u) i:{a..b⁻}. e(i))
. | |  =
. | |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. | | by Decide: b = c
. | |
. | |\
. | | 15. b = c
. | | ⊢  (Iter(f;u) i:{a..b⁻}. e(i))
. | | |  =
. | | |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. | | | by Rewrite by b = c
. | | |    THEN
. | | |    Rewrite by
. | | |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
. | | |          b≤a ⇒ (Iter(f;u) i:{a..b⁻}. e(i)) = u
. | | |    {iter_via_intseg_null}
. | | |    Using:[a:= c | b:= c]
. | | |
. | | ⊢  (Iter(f;u) i:{a..c⁻}. e(i))
. | |    =
. | |    f((Iter(f;u) i:{a..c⁻}. e(i)),u)
. | |   by Inst: is_ident(A; f; u) Using:[Iter(f;u) i:{a..c⁻}. e(i)]
. | |
. | .\
. | . 15. ¬b = c
. | . ⊢  (Iter(f;u) i:{a..b⁻}. e(i))
. | . |  =
. | . |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. | . | by Rewrite by
. | . |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
. | . |          a<b
. | . |          ⇒
. | . |          (Iter(f;u) i:{a..b⁻}. e(i))
. | . |          =
. | . |          f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. | . |    {iter_via_intseg_split_last}
. | . |    Using:[a:= a | b:= b]
. | . |
. | . ⊢  f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. | . |  =
. | . |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. | . | by Rewrite by
. | . |    Thm*  ∀f:(A→A→A), u:A, a,b:ℤ, e:({a..b⁻}→A).
. | . |          a<b
. | . |          ⇒
. | . |          (Iter(f;u) i:{a..b⁻}. e(i))
. | . |          =
. | . |          f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. | . |    {iter_via_intseg_split_last}
. | . |    Using:[a:= c | b:= b]
. | . |
. | . ⊢  f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. | . |  =
. | . |  f
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. | . |  ((Iter(f;u) i:{a..c⁻}. e(i))
. | . |  ,f((Iter(f;u) i:{c..b-1⁻}. e(i)),e(b-1)))
. | . | by Rewrite by is_assoc_sep(A; f)
. | . |
. | . ⊢  f((Iter(f;u) i:{a..b-1⁻}. e(i)),e(b-1))
. | . |  =
. | . |  f
. | . |  (f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b-1⁻}. e(i))
. | . |  ,e(b-1))  by 2 Times Analyze
. | . |
. | . ⊢  (Iter(f;u) i:{a..b-1⁻}. e(i))
. | .    =
. | .    f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b-1⁻}. e(i))
. | .   by BackThru: Hyp:11 Using:[k-1]
. |
. .\
. . 9. a<c
. . |  ⇒
. . |  (∀k:ℕ, b:ℤ.
. . |   b-c = k
. . |   ⇒
. . |   (∀e:({a..b⁻}→A).
. . |    (Iter(f;u) i:{a..b⁻}. e(i))
. . |    =
. . |    f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))))
. . ⊢  ∀b:ℤ, e:({a..b⁻}→A).
. . |  a≤c
. . |  ⇒
. . |  c≤b
. . |  ⇒
. . |  (Iter(f;u) i:{a..b⁻}. e(i))
. . |  =
. . |  f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. . | by Auto
. . |
. . 10. b : ℤ
. . 11. e : {a..b⁻}→A

. . 12. a≤c
. . 13. c≤b
. . ⊢  (Iter(f;u) i:{a..b⁻}. e(i))
. .    =
. .    f((Iter(f;u) i:{a..c⁻}. e(i)),Iter(f;u) i:{c..b⁻}. e(i))
. .   by BackThru: Hyp:9 Using:[b-c]
END
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