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sublist_transitivity  THM

⊢  Discrete{T}
|  ⇒
|  (∀eq1,eq2,eq3:{T≡₂}, L1,L2,L3:T List.
|   eq1 = eq2 ⇒ eq2 = eq3 ⇒ L1(⊆eq1)L2 ⇒ L2(⊆eq2)L3 ⇒ L1(⊆eq3)L3)
| by Analyze 0 THEN Analyze 0 THEN Analyze 0 THEN Equivalence -1
|    THEN
|    Analyze 0
|    THEN
|    Equivalence -1
|    THEN
|    Analyze 0
|    THEN
|    Equivalence -1
|    THEN
|    UnivCD
|
1. T : Type
2. Discrete{T}
3. eq1 : {T≡₂}
4. eq1 ∈ T→T→𝔹
5. ∀x:T. eq1(x,x)
6. ∀x,y:T. eq1(x,y) ⟺ eq1(y,x)
7. ∀x,y,z:T. eq1(x,y) ⇒ eq1(y,z) ⇒ eq1(x,z)
8. eq2 : {T≡₂}
9. eq2 ∈ T→T→𝔹
10. ∀x:T. eq2(x,x)
11. ∀x,y:T. eq2(x,y) ⟺ eq2(y,x)
12. ∀x,y,z:T. eq2(x,y) ⇒ eq2(y,z) ⇒ eq2(x,z)
13. eq3 : {T≡₂}
14. eq3 ∈ T→T→𝔹
15. ∀x:T. eq3(x,x)
16. ∀x,y:T. eq3(x,y) ⟺ eq3(y,x)
17. ∀x,y,z:T. eq3(x,y) ⇒ eq3(y,z) ⇒ eq3(x,z)
18. L1 : T List
19. L2 : T List
20. L3 : T List
21. eq1 = eq2
22. eq2 = eq3
23. L1(⊆eq1)L2
24. L2(⊆eq2)L3
⊢  L1(⊆eq3)L3
| by HypSubst 21 23 THENA Equivalence -1 THEN HypSubst 22 23
|    THENA
|    Equivalence -1
|    THEN
|    HypSubst 22 24
|    THENA
|    Equivalence -1
|
23. L1(⊆eq3)L2
24. L2(⊆eq3)L3
⊢  L1(⊆eq3)L3
| by RenameVar ‘eq’ 13
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|    THEN
|    OnHyps [22;21;12;11;10;9;8;7;6;5;4;3] (λi.Thin i)
|
3. eq : {T≡₂}
4. eq ∈ T→T→𝔹
5. ∀x:T. eq(x,x)
6. ∀x,y:T. eq(x,y) ⟺ eq(y,x)
7. ∀x,y,z:T. eq(x,y) ⇒ eq(y,z) ⇒ eq(x,z)
8. L1 : T List
9. L2 : T List
10. L3 : T List
11. L1(⊆eq)L2
12. L2(⊆eq)L3
⊢  L1(⊆eq)L3  by OnHyps [11;12;0] (λi.Unfold ‘sublist‘ i)
|
11. ∀x∈L1.x(∈eq) L2
12. ∀x∈L2.x(∈eq) L3
⊢  ∀x∈L1.x(∈eq) L3
| by FwdThru
|    Thm* Discrete{T} ⇒ (∃f:{T=₂}. True)
|    {discrete_implies_discrete_equality}
|    [2]
|    THEN
|    Analyze -1
|    THEN
|    Thin -1
|    THEN
|    DiscreteEq -1
|
13. f : {T=₂}
14. f ∈ T→T→𝔹
15. ∀x,y:T. f(x,y) ⟺ x = y
⊢  ∀x∈L1.x(∈eq) L3
| by Using [‘eq’,f]
|    (BackThru
|     Thm* ∀eq:{T=₂}, P:(T→Prop), L:T List.
|          ∀x∈L.P(x) ⟺ (∀z:T. z(∈eq) L ⇒ P(z))
|     {list_all_is_member_lemma})
|
⊢  ∀z:T. z(∈f) L1 ⇒ z(∈eq) L3  by UnivCD
|
16. z : T
17. z(∈f) L1
⊢  z(∈eq) L3
| by Inst
|    Thm* ∀eq:{T=₂}, P:(T→Prop), L:T List.
|         ∀x∈L.P(x) ⟺ (∀z:T. z(∈eq) L ⇒ P(z))

|    {list_all_is_member_lemma}
|    [T;f;λz.z(∈eq) L2;L1]
|    THEN
|    ReduceSOAps -1
|    THEN
|    Analyze -1
|    THEN
|    Thin -1
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|     THEN
|     Inst
|     Thm* ∀eq:{T=₂}, P:(T→Prop), L:T List.
|          ∀x∈L.P(x) ⟺ (∀z:T. z(∈eq) L ⇒ P(z))
|     {list_all_is_member_lemma}
|     [T;f;λz.z(∈eq) L3;L2]
|     THEN
|     ReduceSOAps -1
|     THEN
|     Analyze -1
|     THEN
|     Thin -1
|
18. ∀x∈L1.x(∈eq) L2 ⇒ (∀z:T. z(∈f) L1 ⇒ z(∈eq) L2)
19. ∀x∈L2.x(∈eq) L3 ⇒ (∀z:T. z(∈f) L2 ⇒ z(∈eq) L3)
⊢  z(∈eq) L3
| by RenameVar ‘zz’ 12 THEN With zz (Analyze -1)
|     THEN
|     RenameVar ‘xx’ 11
|     THEN
|     Witness18 xx
|     THEN
|     Thin 12
|     THEN
|     Thin 11
|
11. f : {T=₂}
12. f ∈ T→T→𝔹
13. ∀x,y:T. f(x,y) ⟺ x = y
14. z : T
15. z(∈f) L1
16. ∀z:T. z(∈f) L2 ⇒ z(∈eq) L3
17. ∀z:T. z(∈f) L1 ⇒ z(∈eq) L2
⊢  z(∈eq) L3  by Witness17 z THEN FwdThru -1 [15] THEN Thin 17
|
17. z(∈eq) L2
⊢  z(∈eq) L3  by ListInd 9
|
|\
| 15. z(∈f) L1
| ⊢  (∀z:T. z(∈f) nil ⇒ z(∈eq) L3) ⇒ z(∈eq) nil ⇒ z(∈eq) L3
|   by Rewrite (HigherC is_member_unrollC THENC HigherC assert_evalC)
|      0
|
.\
. 16. u : T
. 17. v : T List
. 18. (∀z:T. z(∈f) v ⇒ z(∈eq) L3) ⇒ z(∈eq) v ⇒ z(∈eq) L3
. ⊢  (∀z:T. z(∈f) (u.v) ⇒ z(∈eq) L3) ⇒ z(∈eq) (u.v) ⇒ z(∈eq) L3
. | by UnivCD
. |
. 19. ∀z:T. z(∈f) (u.v) ⇒ z(∈eq) L3
. 20. z(∈eq) (u.v)
. ⊢  z(∈eq) L3
. | by Rewrite (HigherC is_member_unrollC) -1 THEN MoveToConcl -1
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. |
. 19. ∀z:T. z(∈f) (u.v) ⇒ z(∈eq) L3
. ⊢  if eq(z,u)→ true₂ else z(∈eq) v fi ⇒ z(∈eq) L3
. | by GenConcl (eq(z,u) = b) THEN BoolInd -2
. |    THEN
. |    Rewrite
. |    (HigherC ifthenelse_evalC THENC TryC (HigherC assert_evalC))
. |    0
. |    THENL
. |    [Auto;Analyze 0]
. |
. |\
. | 20. eq(z,u) = true₂
. | 21. True
. | ⊢  z(∈eq) L3  by Witness19 u
. | |
. | 19. eq(z,u) = true₂
. | 20. True
. | 21. u(∈f) (u.v) ⇒ u(∈eq) L3
. | ⊢  z(∈eq) L3  by Rewrite (HigherC is_member_unrollC) 21
. | |
. | 21. if f(u,u)→ true₂ else u(∈f) v fi ⇒ u(∈eq) L3
. | ⊢  z(∈eq) L3  by Assert (f(u,u))
. | |
. | |\
. | | .....assertion .....
. | | ⊢  f(u,u)
. | | | by Witness13 u THEN With u (Analyze -1) THEN Analyze -1
. | | |
. | | 13. z : T
. | | 14. z(∈f) L1
. | | 15. u : T
. | | 16. v : T List
. | | 17. (∀z:T. z(∈f) v ⇒ z(∈eq) L3) ⇒ z(∈eq) v ⇒ z(∈eq) L3
. | | 18. eq(z,u) = true₂
. | | 19. True
. | | 20. if f(u,u)→ true₂ else u(∈f) v fi ⇒ u(∈eq) L3
. | | 21. f(u,u) ⇒ u = u
. | | 22. (f(u,u)) ⇐ (u = u)
. | | ⊢  f(u,u)  by BackThru 22
. | |
. | .\
. | . 22. f(u,u)
. | . ⊢  z(∈eq) L3
. | . | by FwdThru Thm* ∀P:𝔹. P ⟺ P = true₂    {assert_iff_btrue}
. | . |    [22]
. | . |
. | . 23. f(u,u) = true₂
. | . ⊢  z(∈eq) L3  by HypSubst 23 21
. | . |
. | . 21. if true₂→ true₂ else u(∈f) v fi ⇒ u(∈eq) L3
. | . 22. f(u,u)
. | . 23. f(u,u) = true₂
. | . ⊢  z(∈eq) L3
. | . | by Rewrite
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. | . |    (HigherC ifthenelse_evalC THENC HigherC assert_evalC)
. | . |    21
. | . |    THEN
. | . |    Analyze 21
. | . |    THENA
. | . |    Trivial
. | . |
. | . 21. f(u,u)
. | . 22. f(u,u) = true₂
. | . 23. u(∈eq) L3
. | . ⊢  z(∈eq) L3  by Thin 18 THEN ListInd 10
. | . |
. | . |\
. | . | 18. eq(z,u) = true₂
. | . | 19. True
. | . | 20. f(u,u)
. | . | 21. f(u,u) = true₂
. | . | ⊢  u(∈eq) nil ⇒ z(∈eq) nil
. | . |   by Rewrite
. | . |      (HigherC is_member_unrollC THENC HigherC assert_evalC)
. | . |      0
. | . |
. | . .\
. | . . 18. eq(z,u) = true₂
. | . . 19. True
. | . . 20. f(u,u)
. | . . 21. f(u,u) = true₂
. | . . 22. u1 : T
. | . . 23. v1 : T List
. | . . 24. u(∈eq) v1 ⇒ z(∈eq) v1
. | . . ⊢  u(∈eq) (u1.v1) ⇒ z(∈eq) (u1.v1)
. | . . | by Rewrite (HigherC is_member_unrollC) 0
. | . . |
. | . . ⊢  if eq(u,u1)→ true₂ else u(∈eq) v1 fi
. | . . |  ⇒
. | . . |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | by FwdThru Thm* ∀P:𝔹. P ⟺ P = true₂    {assert_iff_btrue}
. | . . |    [18]
. | . . |
. | . . 25. eq(z,u)
. | . . ⊢  if eq(u,u1)→ true₂ else u(∈eq) v1 fi
. | . . |  ⇒
. | . . |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | by GenConcl (eq(u,u1) = b1) THEN BoolInd -2
. | . . |
. | . . |\
. | . . | 26. eq(u,u1) = true₂
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
. | . . | |  ⇒
. | . . | |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | | by FwdThru Thm* ∀P:𝔹. P ⟺ P = true₂    {assert_iff_btrue}
. | . . | |    [26]
. | . . | |
. | . . | 27. eq(u,u1)
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
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. | . . | |  ⇒
. | . . | |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | | by Witness7 z THEN With u (Analyze -1)
. | . . | |    THEN
. | . . | |    With u1 (Analyze -1)
. | . . | |
. | . . | 7. L1 : T List
. | . . | 8. T List
. | . . | 9. T List
. | . . | 10. f : {T=₂}
. | . . | 11. f ∈ T→T→𝔹
. | . . | 12. ∀x,y:T. f(x,y) ⟺ x = y
. | . . | 13. z : T
. | . . | 14. z(∈f) L1
. | . . | 15. u : T
. | . . | 16. T List
. | . . | 17. eq(z,u) = true₂
. | . . | 18. True
. | . . | 19. f(u,u)
. | . . | 20. f(u,u) = true₂
. | . . | 21. u1 : T
. | . . | 22. v1 : T List
. | . . | 23. u(∈eq) v1 ⇒ z(∈eq) v1
. | . . | 24. eq(z,u)
. | . . | 25. eq(u,u1) = true₂
. | . . | 26. eq(u,u1)
. | . . | 27. eq(z,u) ⇒ eq(u,u1) ⇒ eq(z,u1)
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
. | . . | |  ⇒
. | . . | |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | | by FwdThru -1 [24;26]
. | . . | |
. | . . | 28. eq(z,u1)
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
. | . . | |  ⇒
. | . . | |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . | | by FwdThru Thm* ∀P:𝔹. P ⟺ P = true₂    {assert_iff_btrue}
. | . . | |    [28]
. | . . | |
. | . . | 29. eq(z,u1) = true₂
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
. | . . | |  ⇒
. | . . | |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi  by HypSubst 29 0
. | . . | |
. | . . | ⊢  if true₂→ true₂ else u(∈eq) v1 fi
. | . . |    ⇒
. | . . |    if true₂→ true₂ else z(∈eq) v1 fi
. | . . |   by Rewrite
. | . . |     (HigherC ifthenelse_evalC THENC HigherC assert_evalC)
. | . . |     0
. | . . |
. | . . .\
. | . . . 26. eq(u,u1) = false₂
. | . . . ⊢  if false₂→ true₂ else u(∈eq) v1 fi
. | . . . |  ⇒
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. | . . . |  if eq(z,u1)→ true₂ else z(∈eq) v1 fi
. | . . . | by GenConcl (eq(z,u1) = b) THEN BoolInd -2
. | . . . |    THEN
. | . . . |    Rewrite
. | . . . |    (HigherC ifthenelse_evalC
. | . . . |     THENC
. | . . . |     TryC (HigherC assert_evalC))
. | . . . |    0
. | . . . |
. | . . . |\
. | . . . | 27. eq(z,u1) = true₂
. | . . . | ⊢  u(∈eq) v1 ⇒ True  by Auto
. | . . . |
. | . . . .\
. | . . . . 27. eq(z,u1) = false₂
. | . . . . ⊢  u(∈eq) v1 ⇒ z(∈eq) v1  by Trivial
. |
. .\
. . 20. eq(z,u) = false₂
. . 21. z(∈eq) v
. . ⊢  z(∈eq) L3  by BackThru 18
. . |
. . |\
. . | ⊢  ∀z:T. z(∈f) v ⇒ z(∈eq) L3
. . | | by Thin 18 THEN Rewrite (HigherC is_member_unrollC) 18
. . | |
. . | 18. ∀z:T. if f(z,u)→ true₂ else z(∈f) v fi ⇒ z(∈eq) L3
. . | 19. eq(z,u) = false₂
. . | 20. z(∈eq) v
. . | ⊢  ∀z:T. z(∈f) v ⇒ z(∈eq) L3  by UnivCD
. . | |
. . | 21. z1 : T
. . | 22. z1(∈f) v
. . | ⊢  z1(∈eq) L3  by Witness18 z1
. . | |
. . | 18. eq(z,u) = false₂
. . | 19. z(∈eq) v
. . | 20. z1 : T
. . | 21. z1(∈f) v
. . | 22. if f(z1,u)→ true₂ else z1(∈f) v fi ⇒ z1(∈eq) L3
. . | ⊢  z1(∈eq) L3
. . | | by MoveToConcl -1 THEN GenConcl (f(z1,u) = b) THEN BoolInd -2
. . | |    THEN
. . | |    Rewrite
. . | |    (HigherC ifthenelse_evalC
. . | |     THENC
. . | |     TryC (HigherC assert_evalC))
. . | |    0
. . | |    THEN
. . | |    UnivCD
. . | |
. . | |\
. . | | 22. f(z1,u) = true₂
. . | | 23. True ⇒ z1(∈eq) L3
. . | | ⊢  z1(∈eq) L3  by Analyze 23 THEN Trivial
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. . | |
. . | .\
. . | . 22. f(z1,u) = false₂
. . | . 23. z1(∈f) v ⇒ z1(∈eq) L3
. . | . ⊢  z1(∈eq) L3  by FwdThru 23 [21] THEN Trivial
. . |
. . .\
. . . ⊢  z(∈eq) v  by Trivial
END
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