```
before-upto  THM

⊢  ∀n:ℕ, x,y:ℕn. x before y ∈ upto(n) ⟺ x<y
| by Unfold ‘l_before‘ 0 THEN Unfold ‘sublist‘ 0 THEN Reduce 0
|    THEN
|    Reduce 0
|    THEN
|    ExRepD
|
|\
| 1. n : ℕ
| 2. x : ℕn
| 3. y : ℕn
| 4. f : ℕ2→ℕ||upto(n)||
| 5. increasing(f;2)
| 6. ∀j:ℕ2. [x; y][j] = upto(n)[(f(j))]
| ⊢  x<y  by InstHyp [0] -1 THEN InstHyp [1] -2 THEN All Reduce
| |
| 1. n : ℕ
| 2. x : ℕn
| 3. y : ℕn
| 4. f : ℕ2→ℕ||upto(n)||
| 5. increasing(f;2)
| 6. ∀j:ℕ2. [x; y][j] = upto(n)[(f(j))]
| 7. x = upto(n)[(f(0))]
| 8. y = upto(n)[(f(1))]
| ⊢  x<y  by RWO Thm* ∀n:ℕ. ||upto(n)|| ~ n    {length_upto} 4
| |
| 1. n : ℕ
| 2. x : ℕn
| 3. y : ℕn
| 4. f : ℕ2→ℕn
| 5. increasing(f;2)
| 6. ∀j:ℕ2. [x; y][j] = upto(n)[(f(j))]
| 7. x = upto(n)[(f(0))]
| 8. y = upto(n)[(f(1))]
| ⊢  x<y
| | by Inst Thm* ∀m:ℕ, n:ℕm. upto(m)[n] = n    {select_upto} [n;f(0)]
| |    THEN
| |    Inst Thm* ∀m:ℕ, n:ℕm. upto(m)[n] = n    {select_upto} [n;f(1)]
| |
| 1. n : ℕ
| 2. x : ℕn
| 3. y : ℕn
| 4. f : ℕ2→ℕn
| 5. increasing(f;2)
| 6. ∀j:ℕ2. [x; y][j] = upto(n)[(f(j))]
| 7. x = upto(n)[(f(0))]
| 8. y = upto(n)[(f(1))]
| 9. upto(n)[(f(0))] = f(0)
| 10. upto(n)[(f(1))] = f(1)
| ⊢  x<y  by Unfold ‘increasing‘ 5 THEN InstHyp [0] 5
|
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. 1. n : ℕ
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. 2. x : ℕn
. 3. y : ℕn
. 4. x<y
. ⊢  ∃f:(ℕ2→ℕ||upto(n)||).
. |  increasing(f;2) & (∀j:ℕ2. [x; y][j] = upto(n)[(f(j))])
. | by RWO Thm* ∀n:ℕ. ||upto(n)|| ~ n    {length_upto} 0
. |    THEN
. |    InstConcl [λi.if i=₂0→ x else y fi]
. |
. |\
. | 1. n : ℕ
. | 2. x : ℕn
. | 3. y : ℕn
. | 4. x<y
. | ⊢  increasing(λi.if i=₂0→ x else y fi;2)
. | |  & (∀j:ℕ2. [x; y][j] = upto(n)[((λi.if i=₂0→ x else y fi)(j))])
. | | by Analyze 0
. | |
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. | | 1. n : ℕ
. | | 2. x : ℕn
. | | 3. y : ℕn
. | | 4. x<y
. | | ⊢  increasing(λi.if i=₂0→ x else y fi;2)
. | |   by Unfold `increasing` 0 THEN Reduce 0 THEN SplitOnConclITE
. | |      THEN
. | |      SplitOnConclITE
. | |
. | .\
. | . 1. n : ℕ
. | . 2. x : ℕn
. | . 3. y : ℕn
. | . 4. x<y
. | . ⊢  ∀j:ℕ2. [x; y][j] = upto(n)[((λi.if i=₂0→ x else y fi)(j))]
. | . | by Reduce 0
. | . |
. | . 1. n : ℕ
. | . 2. x : ℕn
. | . 3. y : ℕn
. | . 4. x<y
. | . ⊢  ∀j:ℕ2. [x; y][j] = upto(n)[if j=₂0→ x else y fi]
. | . | by Auto THEN IntSegCases -1 THEN Reduce 0
. | . |
. | . |\
. | . | 1. n : ℕ
. | . | 2. x : ℕn
. | . | 3. y : ℕn
. | . | 4. x<y
. | . | 5. j : ℕ2
. | . | 6. j = 0
. | . | ⊢  x = upto(n)[x]
. | . |   by Inst Thm* ∀m:ℕ, n:ℕm. upto(m)[n] = n    {select_upto}
. | . |      [n;x]
. | . |
. | . .\
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. | . . 1. n : ℕ
. | . . 2. x : ℕn
. | . . 3. y : ℕn
. | . . 4. x<y
. | . . 5. j : ℕ2
. | . . 6. ¬j = 0
. | . . 7. j = 1
. | . . ⊢  y = upto(n)[y]
. | . .   by Inst Thm* ∀m:ℕ, n:ℕm. upto(m)[n] = n    {select_upto}
. | . .      [n;y]
. |
. .\
. . 1. n : ℕ
. . 2. x : ℕn
. . 3. y : ℕn
. . 4. x<y
. . 5. f : ℕ2→ℕn
. . 6. j : ℕ2
. . ⊢  f(j)<||upto(n)||
. .   by RWO Thm* ∀n:ℕ. ||upto(n)|| ~ n    {length_upto} 0
END
```