```
mu-property  THM

⊢  ∀f:(ℕ→𝔹). (∃n:ℕ. f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| by Assert
|    (∀b:ℕ, f:(ℕ→𝔹).
|     (∃n:ℕ. n≤b & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i)))
|
|\
| .....assertion .....
| ⊢  ∀b:ℕ, f:(ℕ→𝔹).
| |  (∃n:ℕ. n≤b & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| | by InductionOnNat THEN RepeatFor 2 (Analyze 0) THEN ExRepD
| |    THEN
| |    RecUnfold ‘mu‘ 0
| |    THEN
| |    SplitOnConclITE
| |
| |\
| | .....truecase .....
| | 1. f : ℕ→𝔹
| | 2. n : ℕ
| | 3. n≤0
| | 4. f(n)
| | 5. f(0)
| | ⊢  f(0) & (∀i:ℕ. i<0 ⇒ ¬f(i))  by Auto
| |
| |\
| | .....falsecase .....
| | 1. f : ℕ→𝔹
| | 2. n : ℕ
| | 3. n≤0
| | 4. f(n)
| | 5. ¬f(0)
| | ⊢  f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1))+1 ⇒ ¬f(i))
| |   by Subst’ (n = 0) -2
| |
| |\
| | .....truecase .....
| | 1. b : ℤ
| | 2. 0<b
| | 3. ∀f:(ℕ→𝔹).
| |    (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| | 4. f : ℕ→𝔹
| | 5. n : ℕ
| | 6. n≤b
| | 7. f(n)
| | 8. f(0)
| | ⊢  f(0) & (∀i:ℕ. i<0 ⇒ ¬f(i))  by Auto
| |
| .\
| . .....falsecase .....
| . 1. b : ℤ
| . 2. 0<b
| . 3. ∀f:(ℕ→𝔹).
| . |  (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
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| . 4. f : ℕ→𝔹
| . 5. n : ℕ
| . 6. n≤b
| . 7. f(n)
| . 8. ¬f(0)
| . ⊢  f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1))+1 ⇒ ¬f(i))
| . | by CaseNat 0 ‘n’
| . |
| . |\
| . | 1. b : ℤ
| . | 2. 0<b
| . | 3. ∀f:(ℕ→𝔹).
| . |      (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . | 4. f : ℕ→𝔹
| . | 5. n : ℕ
| . | 6. n≤b
| . | 7. f(n)
| . | 8. ¬f(0)
| . | 9. n = 0
| . | ⊢  f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1))+1 ⇒ ¬f(i))
| . |   by HypSubst’ -1 -3
| . |
| . .\
| . . 1. b : ℤ
| . . 2. 0<b
| . . 3. ∀f:(ℕ→𝔹).
| . . |  (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . 4. f : ℕ→𝔹
| . . 5. n : ℕ
| . . 6. n≤b
| . . 7. f(n)
| . . 8. ¬f(0)
| . . 9. ¬n = 0
| . . ⊢  f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1))+1 ⇒ ¬f(i))
| . . | by InstHyp [λx.f(x+1)] 3 THEN All Reduce
| . . |
| . . |\
| . . | 1. b : ℤ
| . . | 2. 0<b
| . . | 3. ∀f:(ℕ→𝔹).
| . . | |  (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . | 4. f : ℕ→𝔹
| . . | 5. n : ℕ
| . . | 6. n≤b
| . . | 7. f(n)
| . . | 8. ¬f(0)
| . . | 9. ¬n = 0
| . . | ⊢  ∃n:ℕ. n≤b-1 & f(n+1)  by InstConcl [n-1]
| . . | |
| . . | 1. b : ℤ
| . . | 2. 0<b
| . . | 3. ∀f:(ℕ→𝔹).
| . . |      (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . | 4. f : ℕ→𝔹
| . . | 5. n : ℕ
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| . . | 6. n≤b
| . . | 7. f(n)
| . . | 8. ¬f(0)
| . . | 9. ¬n = 0
| . . | ⊢  f(n-1+1)  by ArithSimp 0
| . . |
| . . .\
| . . . 1. b : ℤ
| . . . 2. 0<b
| . . . 3. ∀f:(ℕ→𝔹).
| . . . |  (∃n:ℕ. n≤b-1 & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . 4. f : ℕ→𝔹
| . . . 5. n : ℕ
| . . . 6. n≤b
| . . . 7. f(n)
| . . . 8. ¬f(0)
| . . . 9. ¬n = 0
| . . . 10. f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1))
| . . . ⊢  f(mu(λx.f(x+1))+1) & (∀i:ℕ. i<mu(λx.f(x+1))+1 ⇒ ¬f(i))
| . . . | by Auto THEN All Reduce
| . . . |
| . . . |\
| . . . | 1. b : ℤ
| . . . | 2. 0<b
| . . . | 3. ∀f:(ℕ→𝔹).
| . . . | |  (∃n:ℕ. n≤b-1 & f(n))
| . . . | |  ⇒
| . . . | |  f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . | 4. f : ℕ→𝔹
| . . . | 5. n : ℕ
| . . . | 6. n≤b
| . . . | 7. f(n)
| . . . | 8. ¬f(0)
| . . . | 9. ¬n = 0
| . . . | 10. f(mu(λx.f(x+1))+1)
| . . . | 11. ∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1)
| . . . | 12. i : ℕ
| . . . | 13. i<mu(λx.f(x+1))+1
| . . . | ⊢  ¬f(i)  by CaseNat 0 'i'
| . . . | |
| . . . | 1. b : ℤ
| . . . | 2. 0<b
| . . . | 3. ∀f:(ℕ→𝔹).
| . . . | |  (∃n:ℕ. n≤b-1 & f(n))
| . . . | |  ⇒
| . . . | |  f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . | 4. f : ℕ→𝔹
| . . . | 5. n : ℕ
| . . . | 6. n≤b
| . . . | 7. f(n)
| . . . | 8. ¬f(0)
| . . . | 9. ¬n = 0
| . . . | 10. f(mu(λx.f(x+1))+1)
| . . . | 11. ∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1)
| . . . | 12. i : ℕ
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| . . . | 13. i<mu(λx.f(x+1))+1
| . . . | 14. ¬i = 0
| . . . | ⊢  ¬f(i)  by InstHyp [i-1] -4
| . . . | |
| . . . | |\
| . . . | | 1. b : ℤ
| . . . | | 2. 0<b
| . . . | | 3. ∀f:(ℕ→𝔹).
| . . . | |    (∃n:ℕ. n≤b-1 & f(n))
| . . . | |    ⇒
| . . . | |    f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . | | 4. f : ℕ→𝔹
| . . . | | 5. n : ℕ
| . . . | | 6. n≤b
| . . . | | 7. f(n)
| . . . | | 8. ¬f(0)
| . . . | | 9. ¬n = 0
| . . . | | 10. f(mu(λx.f(x+1))+1)
| . . . | | 11. ∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1)
| . . . | | 12. i : ℕ
| . . . | | 13. i<mu(λx.f(x+1))+1
| . . . | | 14. ¬i = 0
| . . . | | ⊢  ∃n:ℕ. (λx.f(x+1))(n)
| . . . | |   by (Reduce 0 THEN InstConcl [n-1]) THEN (ArithSimp 0)
| . . . | |
| . . . | .\
| . . . | . 1. b : ℤ
| . . . | . 2. 0<b
| . . . | . 3. ∀f:(ℕ→𝔹).
| . . . | .    (∃n:ℕ. n≤b-1 & f(n))
| . . . | .    ⇒
| . . . | .    f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . | . 4. f : ℕ→𝔹
| . . . | . 5. n : ℕ
| . . . | . 6. n≤b
| . . . | . 7. f(n)
| . . . | . 8. ¬f(0)
| . . . | . 9. ¬n = 0
| . . . | . 10. f(mu(λx.f(x+1))+1)
| . . . | . 11. ∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1)
| . . . | . 12. i : ℕ
| . . . | . 13. i<mu(λx.f(x+1))+1
| . . . | . 14. ¬i = 0
| . . . | . 15. ¬f(i-1+1)
| . . . | . ⊢  ¬f(i)  by ArithSimp -1
| . . . |
| . . . .\
| . . . . 1. b : ℤ
| . . . . 2. 0<b
| . . . . 3. ∀f:(ℕ→𝔹).
| . . . .    (∃n:ℕ. n≤b-1 & f(n))
| . . . .    ⇒
| . . . .    f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
| . . . . 4. f : ℕ→𝔹
| . . . . 5. n : ℕ
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| . . . . 6. n≤b
| . . . . 7. f(n)
| . . . . 8. ¬f(0)
| . . . . 9. ¬n = 0
| . . . . 10. f(mu(λx.f(x+1))+1)
| . . . . 11. ∀i:ℕ. i<mu(λx.f(x+1)) ⇒ ¬f(i+1)
| . . . . 12. ℕ
| . . . . ⊢  ∃n:ℕ. f(n+1)  by (InstConcl [n-1]) THEN (ArithSimp 0)
|
.\
. 1. ∀b:ℕ, f:(ℕ→𝔹).
. |  (∃n:ℕ. n≤b & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
. ⊢  ∀f:(ℕ→𝔹). (∃n:ℕ. f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
. | by Auto THEN ExRepD THEN InstHyp [n;f] 1
. |
. |\
. | .....antecedent .....
. | 1. ∀b:ℕ, f:(ℕ→𝔹).
. |    (∃n:ℕ. n≤b & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
. | 2. f : ℕ→𝔹
. | 3. n : ℕ
. | 4. f(n)
. | ⊢  ∃n@0:ℕ. n@0≤n & f(n@0)  by InstConcl [n]
. |
. .\
. . 1. ∀b:ℕ, f:(ℕ→𝔹).
. .    (∃n:ℕ. n≤b & f(n)) ⇒ f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
. . 2. f : ℕ→𝔹
. . 3. n : ℕ
. . 4. f(n)
. . 5. f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))
. . ⊢  f(mu(f)) & (∀i:ℕ. i<mu(f) ⇒ ¬f(i))  by Trivial
END
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