```
ring-list  THM

⊢  ∀R:(Id→𝔹), in,out:(|R|→IdLnk).
|  ring(R;in;out) ⇒ (∃L:|R| List. 0<||L|| & (∀i:|R|. (i ∈ L)))
| by Auto THEN Unfold `ring` -1 THEN ExRepD THEN RenameVar `i' -1
|
1. R : Id→𝔹
2. in : |R|→IdLnk
3. out : |R|→IdLnk
4. ∀i:|R|.
|  R(source(in(i))) & R(destination(out(i)))
|  & source(out(i)) = i
|    & destination(in(i)) = i
|    & in(destination(out(i))) = out(i)
|    & out(source(in(i))) = in(i)
5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
6. |R|
⊢  ∃L:|R| List. 0<||L|| & (∀i:|R|. (i ∈ L))
| by InstHyp [i;i] -2 THEN ExRepD
|
1. R : Id→𝔹
2. in : |R|→IdLnk
3. out : |R|→IdLnk
4. ∀i:|R|.
|  R(source(in(i))) & R(destination(out(i)))
|  & source(out(i)) = i
|    & destination(in(i)) = i
|    & in(destination(out(i))) = out(i)
|    & out(source(in(i))) = in(i)
5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
6. i : |R|
7. k : ℕ+
8. λx.destination(out(x))^k(i) = i
⊢  ∃L:|R| List. 0<||L|| & (∀i:|R|. (i ∈ L))
| by Assert ((λx.destination(out(x))) ∈ |R|→|R|)
|
|\
| .....assertion .....
| 1. R : Id→𝔹
| 2. in : |R|→IdLnk
| 3. out : |R|→IdLnk
| 4. ∀i:|R|.
|    R(source(in(i))) & R(destination(out(i)))
|    & source(out(i)) = i
|      & destination(in(i)) = i
|      & in(destination(out(i))) = out(i)
|      & out(source(in(i))) = in(i)
| 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
| 6. i : |R|
| 7. k : ℕ+
| 8. λx.destination(out(x))^k(i) = i
| ⊢  (λx.destination(out(x))) ∈ |R|→|R|
|   by Auto THEN Analyze THEN InstHyp [x] 4 THEN ExRepD
|
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. 1. R : Id→𝔹
. 2. in : |R|→IdLnk
. 3. out : |R|→IdLnk
. 4. ∀i:|R|.
. |  R(source(in(i))) & R(destination(out(i)))
. |  & source(out(i)) = i
. |    & destination(in(i)) = i
. |    & in(destination(out(i))) = out(i)
. |    & out(source(in(i))) = in(i)
. 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. 6. i : |R|
. 7. k : ℕ+
. 8. λx.destination(out(x))^k(i) = i
. 9. (λx.destination(out(x))) ∈ |R|→|R|
. ⊢  ∃L:|R| List. 0<||L|| & (∀i:|R|. (i ∈ L))
. | by InstConcl [map(λk.λx.destination(out(x))^k(i);upto(k))]
. |
. |\
. | .....subterm .....
. | 1. R : Id→𝔹
. | 2. in : |R|→IdLnk
. | 3. out : |R|→IdLnk
. | 4. ∀i:|R|.
. |    R(source(in(i))) & R(destination(out(i)))
. |    & source(out(i)) = i
. |      & destination(in(i)) = i
. |      & in(destination(out(i))) = out(i)
. |      & out(source(in(i))) = in(i)
. | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. | 6. i : |R|
. | 7. k : ℕ+
. | 8. λx.destination(out(x))^k(i) = i
. | 9. (λx.destination(out(x))) ∈ |R|→|R|
. | 10. k1 : ℕk
. | ⊢  λx.destination(out(x))^k1(i) ∈ |R|
. |   by GenConcl ((λx.destination(out(x))) = f)
. |
. .\
. . 1. R : Id→𝔹
. . 2. in : |R|→IdLnk
. . 3. out : |R|→IdLnk
. . 4. ∀i:|R|.
. . |  R(source(in(i))) & R(destination(out(i)))
. . |  & source(out(i)) = i
. . |    & destination(in(i)) = i
. . |    & in(destination(out(i))) = out(i)
. . |    & out(source(in(i))) = in(i)
. . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . 6. i : |R|
. . 7. k : ℕ+
. . 8. λx.destination(out(x))^k(i) = i
. . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . ⊢  0<||map(λk.λx.destination(out(x))^k(i);upto(k))||
. . |  & (∀i@0:|R|.
. . |     (i@0 ∈ map(λk.λx.destination(out(x))^k(i);upto(k))))
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. . | by Analyze 0
. . |
. . |\
. . | 1. R : Id→𝔹
. . | 2. in : |R|→IdLnk
. . | 3. out : |R|→IdLnk
. . | 4. ∀i:|R|.
. . |    R(source(in(i))) & R(destination(out(i)))
. . |    & source(out(i)) = i
. . |      & destination(in(i)) = i
. . |      & in(destination(out(i))) = out(i)
. . |      & out(source(in(i))) = in(i)
. . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . | 6. i : |R|
. . | 7. k : ℕ+
. . | 8. λx.destination(out(x))^k(i) = i
. . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . | ⊢  0<||map(λk.λx.destination(out(x))^k(i);upto(k))||
. . |   by GenConcl ((λx.destination(out(x))) = f)
. . |      THEN
. . |      RWO
. . |      Thm* ∀f:(A→B), as:A List. ||map(f;as)|| = ||as|| ∈ ℤ
. . |      {map_length}
. . |      0
. . |      THEN
. . |      RWO Thm* ∀n:ℕ. ||upto(n)|| ~ n    {length_upto} 0
. . |
. . .\
. . . 1. R : Id→𝔹
. . . 2. in : |R|→IdLnk
. . . 3. out : |R|→IdLnk
. . . 4. ∀i:|R|.
. . . |  R(source(in(i))) & R(destination(out(i)))
. . . |  & source(out(i)) = i
. . . |    & destination(in(i)) = i
. . . |    & in(destination(out(i))) = out(i)
. . . |    & out(source(in(i))) = in(i)
. . . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . 6. i : |R|
. . . 7. k : ℕ+
. . . 8. λx.destination(out(x))^k(i) = i
. . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . ⊢  ∀i@0:|R|. (i@0 ∈ map(λk.λx.destination(out(x))^k(i);upto(k)))
. . . | by Analyze 0
. . . |    THEN
. . . |    RWO
. . . |    Thm* ∀a:T List, x:T', f:(T→T').
. . . |         (x ∈ map(f;a)) ⟺ (∃y:T. (y ∈ a) & x = f(y))
. . . |    {member_map}
. . . |    0
. . . |
. . . |\
. . . | .....subterm .....
. . . | 1. R : Id→𝔹
. . . | 2. in : |R|→IdLnk
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. . . | 3. out : |R|→IdLnk
. . . | 4. ∀i:|R|.
. . . |    R(source(in(i))) & R(destination(out(i)))
. . . |    & source(out(i)) = i
. . . |      & destination(in(i)) = i
. . . |      & in(destination(out(i))) = out(i)
. . . |      & out(source(in(i))) = in(i)
. . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . | 6. i : |R|
. . . | 7. k : ℕ+
. . . | 8. λx.destination(out(x))^k(i) = i
. . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . | 10. |R|
. . . | 11. k1 : ℕk
. . . | ⊢  λx.destination(out(x))^k1(i) ∈ |R|
. . . |   by GenConcl ((λx.destination(out(x))) = f)
. . . |
. . . .\
. . . . 1. R : Id→𝔹
. . . . 2. in : |R|→IdLnk
. . . . 3. out : |R|→IdLnk
. . . . 4. ∀i:|R|.
. . . . |  R(source(in(i))) & R(destination(out(i)))
. . . . |  & source(out(i)) = i
. . . . |    & destination(in(i)) = i
. . . . |    & in(destination(out(i))) = out(i)
. . . . |    & out(source(in(i))) = in(i)
. . . . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . 6. i : |R|
. . . . 7. k : ℕ+
. . . . 8. λx.destination(out(x))^k(i) = i
. . . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . 10. i@0 : |R|
. . . . ⊢  ∃y:ℕk.
. . . . |  (y ∈ upto(k)) & i@0 = (λk.λx.destination(out(x))^k(i))(y)
. . . . | by Reduce 0 THEN Decide (i@0 = i)
. . . . |

. . . . |\
. . . . | .....wf .....
. . . . | 1. R : Id→𝔹
. . . . | 2. in : |R|→IdLnk
. . . . | 3. out : |R|→IdLnk
. . . . | 4. ∀i:|R|.
. . . . |    R(source(in(i))) & R(destination(out(i)))
. . . . |    & source(out(i)) = i
. . . . |      & destination(in(i)) = i
. . . . |      & in(destination(out(i))) = out(i)
. . . . |      & out(source(in(i))) = in(i)
. . . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . | 6. i : |R|
. . . . | 7. k : ℕ+
. . . . | 8. λx.destination(out(x))^k(i) = i
. . . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . | 10. i@0 : |R|
. . . . | ⊢  i@0 ∈ Id  by Analyze -1
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. . . . |
. . . . |\
. . . . | 1. R : Id→𝔹
. . . . | 2. in : |R|→IdLnk
. . . . | 3. out : |R|→IdLnk
. . . . | 4. ∀i:|R|.
. . . . | |  R(source(in(i))) & R(destination(out(i)))
. . . . | |  & source(out(i)) = i
. . . . | |    & destination(in(i)) = i
. . . . | |    & in(destination(out(i))) = out(i)
. . . . | |    & out(source(in(i))) = in(i)
. . . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . | 6. i : |R|
. . . . | 7. k : ℕ+
. . . . | 8. λx.destination(out(x))^k(i) = i
. . . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . | 10. i@0 : |R|
. . . . | 11. i@0 = i
. . . . | ⊢  ∃y:ℕk. (y ∈ upto(k)) & i@0 = λx.destination(out(x))^y(i)
. . . . | | by InstConcl [0]
. . . . | |
. . . . | |\
. . . . | | 1. R : Id→𝔹
. . . . | | 2. in : |R|→IdLnk
. . . . | | 3. out : |R|→IdLnk
. . . . | | 4. ∀i:|R|.
. . . . | |    R(source(in(i))) & R(destination(out(i)))
. . . . | |    & source(out(i)) = i
. . . . | |      & destination(in(i)) = i
. . . . | |      & in(destination(out(i))) = out(i)
. . . . | |      & out(source(in(i))) = in(i)
. . . . | | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . | | 6. i : |R|
. . . . | | 7. k : ℕ+
. . . . | | 8. λx.destination(out(x))^k(i) = i
. . . . | | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . | | 10. i@0 : |R|
. . . . | | 11. i@0 = i
. . . . | | ⊢  (0 ∈ upto(k))
. . . . | |   by BackThru
. . . . | |      Thm* ∀n,i:ℕ. i<n ⇒ (i ∈ upto(n))    {member_upto2}
. . . . | |
. . . . | |\
. . . . | | 1. R : Id→𝔹
. . . . | | 2. in : |R|→IdLnk
. . . . | | 3. out : |R|→IdLnk
. . . . | | 4. ∀i:|R|.
. . . . | |    R(source(in(i))) & R(destination(out(i)))
. . . . | |    & source(out(i)) = i
. . . . | |      & destination(in(i)) = i
. . . . | |      & in(destination(out(i))) = out(i)
. . . . | |      & out(source(in(i))) = in(i)
. . . . | | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . | | 6. i : |R|
. . . . | | 7. k : ℕ+
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. . . . | | 8. λx.destination(out(x))^k(i) = i
. . . . | | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . | | 10. i@0 : |R|
. . . . | | 11. i@0 = i
. . . . | | ⊢  i@0 = λx.destination(out(x))^0(i)
. . . . | |   by Reduce 0 THEN Analyze -2 THEN Analyze
. . . . | |
. . . . | .\
. . . . | . .....subterm .....
. . . . | . 1. R : Id→𝔹
. . . . | . 2. in : |R|→IdLnk
. . . . | . 3. out : |R|→IdLnk
. . . . | . 4. ∀i:|R|.
. . . . | .    R(source(in(i))) & R(destination(out(i)))
. . . . | .    & source(out(i)) = i
. . . . | .      & destination(in(i)) = i
. . . . | .      & in(destination(out(i))) = out(i)
. . . . | .      & out(source(in(i))) = in(i)
. . . . | . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . | . 6. i : |R|
. . . . | . 7. k : ℕ+
. . . . | . 8. λx.destination(out(x))^k(i) = i
. . . . | . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . | . 10. i@0 : |R|
. . . . | . 11. i@0 = i
. . . . | . 12. y : ℕk
. . . . | . ⊢  λx.destination(out(x))^y(i) ∈ |R|
. . . . | .   by GenConcl ((λx.destination(out(x))) = f)
. . . . |
. . . . .\
. . . . . 1. R : Id→𝔹
. . . . . 2. in : |R|→IdLnk
. . . . . 3. out : |R|→IdLnk
. . . . . 4. ∀i:|R|.
. . . . . |  R(source(in(i))) & R(destination(out(i)))
. . . . . |  & source(out(i)) = i
. . . . . |    & destination(in(i)) = i
. . . . . |    & in(destination(out(i))) = out(i)
. . . . . |    & out(source(in(i))) = in(i)
. . . . . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . 6. i : |R|
. . . . . 7. k : ℕ+
. . . . . 8. λx.destination(out(x))^k(i) = i
. . . . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . 10. i@0 : |R|
. . . . . 11. ¬i@0 = i
. . . . . ⊢  ∃y:ℕk. (y ∈ upto(k)) & i@0 = λx.destination(out(x))^y(i)
. . . . . | by Assert
. . . . . |    (∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j)
. . . . . |
. . . . . |\
. . . . . | .....assertion .....
. . . . . | 1. R : Id→𝔹
. . . . . | 2. in : |R|→IdLnk
. . . . . | 3. out : |R|→IdLnk
```

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. . . . . | 4. ∀i:|R|.
. . . . . | |  R(source(in(i))) & R(destination(out(i)))
. . . . . | |  & source(out(i)) = i
. . . . . | |    & destination(in(i)) = i
. . . . . | |    & in(destination(out(i))) = out(i)
. . . . . | |    & out(source(in(i))) = in(i)
. . . . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | 6. i : |R|
. . . . . | 7. k : ℕ+
. . . . . | 8. λx.destination(out(x))^k(i) = i
. . . . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | 10. i@0 : |R|
. . . . . | 11. ¬i@0 = i
. . . . . | ⊢  ∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . | | by Auto
. . . . . | |    THEN
. . . . . | |    Assert
. . . . . | |    (∀n:ℕ.
. . . . . | |     λx.destination(out(x))^n(i) = j
. . . . . | |     ⇒
. . . . . | |     (∃m:ℕk. λx.destination(out(x))^m(i) = j))
. . . . . | |
. . . . . | |\
. . . . . | | .....assertion .....
. . . . . | | 1. R : Id→𝔹
. . . . . | | 2. in : |R|→IdLnk
. . . . . | | 3. out : |R|→IdLnk
. . . . . | | 4. ∀i:|R|.
. . . . . | | |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | |  & source(out(i)) = i
. . . . . | | |    & destination(in(i)) = i
. . . . . | | |    & in(destination(out(i))) = out(i)
. . . . . | | |    & out(source(in(i))) = in(i)
. . . . . | | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | 6. i : |R|
. . . . . | | 7. k : ℕ+
. . . . . | | 8. λx.destination(out(x))^k(i) = i
. . . . . | | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | 10. i@0 : |R|
. . . . . | | 11. ¬i@0 = i
. . . . . | | 12. j : |R|
. . . . . | | ⊢  ∀n:ℕ.
. . . . . | | |  λx.destination(out(x))^n(i) = j
. . . . . | | |  ⇒
. . . . . | | |  (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | by CompleteInductionOnNat THEN Analyze 0
. . . . . | | |
. . . . . | | |\
. . . . . | | | 1. R : Id→𝔹
. . . . . | | | 2. in : |R|→IdLnk
. . . . . | | | 3. out : |R|→IdLnk
. . . . . | | | 4. ∀i:|R|.
. . . . . | | | |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | |  & source(out(i)) = i
. . . . . | | | |    & destination(in(i)) = i
```

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. . . . . | | | |    & in(destination(out(i))) = out(i)
. . . . . | | | |    & out(source(in(i))) = in(i)
. . . . . | | | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | 6. i : |R|
. . . . . | | | 7. k : ℕ+
. . . . . | | | 8. λx.destination(out(x))^k(i) = i
. . . . . | | | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | 10. i@0 : |R|
. . . . . | | | 11. ¬i@0 = i
. . . . . | | | 12. j : |R|
. . . . . | | | 13. n : ℕ
. . . . . | | | 14. ∀n1:ℕ.
. . . . . | | | |   n1<n
. . . . . | | | |   ⇒
. . . . . | | | |   λx.destination(out(x))^n1(i) = j
. . . . . | | | |   ⇒
. . . . . | | | |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | 15. λx.destination(out(x))^n(i) = j
. . . . . | | | ⊢  ∃m:ℕk. λx.destination(out(x))^m(i) = j
. . . . . | | | | by Decide (n<k)
. . . . . | | | |
. . . . . | | | |\
. . . . . | | | | 1. R : Id→𝔹
. . . . . | | | | 2. in : |R|→IdLnk
. . . . . | | | | 3. out : |R|→IdLnk
. . . . . | | | | 4. ∀i:|R|.
. . . . . | | | |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | |    & source(out(i)) = i
. . . . . | | | |      & destination(in(i)) = i
. . . . . | | | |      & in(destination(out(i))) = out(i)
. . . . . | | | |      & out(source(in(i))) = in(i)
. . . . . | | | | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | | 6. i : |R|
. . . . . | | | | 7. k : ℕ+
. . . . . | | | | 8. λx.destination(out(x))^k(i) = i
. . . . . | | | | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | | 10. i@0 : |R|
. . . . . | | | | 11. ¬i@0 = i
. . . . . | | | | 12. j : |R|
. . . . . | | | | 13. n : ℕ
. . . . . | | | | 14. ∀n1:ℕ.
. . . . . | | | |     n1<n
. . . . . | | | |     ⇒
. . . . . | | | |     λx.destination(out(x))^n1(i) = j
. . . . . | | | |     ⇒
. . . . . | | | |     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | | 15. λx.destination(out(x))^n(i) = j
. . . . . | | | | 16. n<k
. . . . . | | | | ⊢  ∃m:ℕk. λx.destination(out(x))^m(i) = j
. . . . . | | | |   by InstConcl [n] THEN Try (Complete Auto)
. . . . . | | | |      THEN
. . . . . | | | |      GenConcl ((λx.destination(out(x))) = f)
. . . . . | | | |      THEN
. . . . . | | | |      DoSubsume
. . . . . | | | |      THEN
```

```
. . . . . | | | |      Analyze 0
. . . . . | | | |      THEN
. . . . . | | | |      Analyze -1
. . . . . | | | |
. . . . . | | | .\
. . . . . | | | . 1. R : Id→𝔹
. . . . . | | | . 2. in : |R|→IdLnk
. . . . . | | | . 3. out : |R|→IdLnk
. . . . . | | | . 4. ∀i:|R|.
. . . . . | | | . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . |  & source(out(i)) = i
. . . . . | | | . |    & destination(in(i)) = i
. . . . . | | | . |    & in(destination(out(i))) = out(i)
. . . . . | | | . |    & out(source(in(i))) = in(i)
. . . . . | | | . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . 6. i : |R|
. . . . . | | | . 7. k : ℕ+
. . . . . | | | . 8. λx.destination(out(x))^k(i) = i
. . . . . | | | . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . 10. i@0 : |R|
. . . . . | | | . 11. ¬i@0 = i
. . . . . | | | . 12. j : |R|
. . . . . | | | . 13. n : ℕ
. . . . . | | | . 14. ∀n1:ℕ.
. . . . . | | | . |   n1<n
. . . . . | | | . |   ⇒
. . . . . | | | . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . |   ⇒
. . . . . | | | . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . 15. λx.destination(out(x))^n(i) = j
. . . . . | | | . 16. ¬n<k
. . . . . | | | . ⊢  ∃m:ℕk. λx.destination(out(x))^m(i) = j
. . . . . | | | . | by InstHyp [n-k] -3
. . . . . | | | . |
. . . . . | | | . .....antecedent .....
. . . . . | | | . 1. R : Id→𝔹
. . . . . | | | . 2. in : |R|→IdLnk
. . . . . | | | . 3. out : |R|→IdLnk
. . . . . | | | . 4. ∀i:|R|.
. . . . . | | | . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . |  & source(out(i)) = i
. . . . . | | | . |    & destination(in(i)) = i
. . . . . | | | . |    & in(destination(out(i))) = out(i)
. . . . . | | | . |    & out(source(in(i))) = in(i)
. . . . . | | | . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . 6. i : |R|
. . . . . | | | . 7. k : ℕ+
. . . . . | | | . 8. λx.destination(out(x))^k(i) = i
. . . . . | | | . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . 10. i@0 : |R|
. . . . . | | | . 11. ¬i@0 = i
. . . . . | | | . 12. j : |R|
. . . . . | | | . 13. n : ℕ
. . . . . | | | . 14. ∀n1:ℕ.
. . . . . | | | . |   n1<n
```

```
. . . . . | | | . |   ⇒
. . . . . | | | . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . |   ⇒
. . . . . | | | . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . 15. λx.destination(out(x))^n(i) = j
. . . . . | | | . 16. ¬n<k
. . . . . | | | . ⊢  λx.destination(out(x))^n-k(i) = j

. . . . . | | | . | by Subst (n ~ (n-k+k)) -2
. . . . . | | | . |
. . . . . | | | . |\
. . . . . | | | . | .....equality .....
. . . . . | | | . | 1. R : Id→𝔹
. . . . . | | | . | 2. in : |R|→IdLnk
. . . . . | | | . | 3. out : |R|→IdLnk
. . . . . | | | . | 4. ∀i:|R|.
. . . . . | | | . |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . |    & source(out(i)) = i
. . . . . | | | . |      & destination(in(i)) = i
. . . . . | | | . |      & in(destination(out(i))) = out(i)
. . . . . | | | . |      & out(source(in(i))) = in(i)
. . . . . | | | . | 5. ∀i,j:|R|.
. . . . . | | | . |    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . | 6. i : |R|
. . . . . | | | . | 7. k : ℕ+
. . . . . | | | . | 8. λx.destination(out(x))^k(i) = i
. . . . . | | | . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . | 10. i@0 : |R|
. . . . . | | | . | 11. ¬i@0 = i
. . . . . | | | . | 12. j : |R|
. . . . . | | | . | 13. n : ℕ
. . . . . | | | . | 14. ∀n1:ℕ.
. . . . . | | | . |     n1<n
. . . . . | | | . |     ⇒
. . . . . | | | . |     λx.destination(out(x))^n1(i) = j
. . . . . | | | . |     ⇒
. . . . . | | | . |     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . | 15. λx.destination(out(x))^n(i) = j
. . . . . | | | . | 16. ¬n<k
. . . . . | | | . | ⊢  n ~ (n-k+k)  by Auto
. . . . . | | | . |
. . . . . | | | . .\
. . . . . | | | . . 1. R : Id→𝔹
. . . . . | | | . . 2. in : |R|→IdLnk
. . . . . | | | . . 3. out : |R|→IdLnk
. . . . . | | | . . 4. ∀i:|R|.
. . . . . | | | . . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . |  & source(out(i)) = i
. . . . . | | | . . |    & destination(in(i)) = i
. . . . . | | | . . |    & in(destination(out(i))) = out(i)
. . . . . | | | . . |    & out(source(in(i))) = in(i)
. . . . . | | | . . 5. ∀i,j:|R|.
. . . . . | | | . . |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . 6. i : |R|
. . . . . | | | . . 7. k : ℕ+
. . . . . | | | . . 8. λx.destination(out(x))^k(i) = i
```

```
. . . . . | | | . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . 10. i@0 : |R|
. . . . . | | | . . 11. ¬i@0 = i
. . . . . | | | . . 12. j : |R|
. . . . . | | | . . 13. n : ℕ
. . . . . | | | . . 14. ∀n1:ℕ.
. . . . . | | | . . |   n1<n
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . 15. λx.destination(out(x))^n-k+k(i) = j
. . . . . | | | . . 16. ¬n<k
. . . . . | | | . . ⊢  λx.destination(out(x))^n-k(i) = j
. . . . . | | | . . | by RWO
. . . . . | | | . . |    Thm* ∀n,m:ℕ, f,i:Top.
. . . . . | | | . . |         (f^n+m(i)) ~ (f^n(f^m(i)))
. . . . . | | | . . |    {fun_exp_add_sq}
. . . . . | | | . . |    -2
. . . . . | | | . . |
. . . . . | | | . . 1. R : Id→𝔹
. . . . . | | | . . 2. in : |R|→IdLnk
. . . . . | | | . . 3. out : |R|→IdLnk
. . . . . | | | . . 4. ∀i:|R|.
. . . . . | | | . . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . |  & source(out(i)) = i
. . . . . | | | . . |    & destination(in(i)) = i
. . . . . | | | . . |    & in(destination(out(i))) = out(i)
. . . . . | | | . . |    & out(source(in(i))) = in(i)
. . . . . | | | . . 5. ∀i,j:|R|.
. . . . . | | | . . |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . 6. i : |R|
. . . . . | | | . . 7. k : ℕ+
. . . . . | | | . . 8. λx.destination(out(x))^k(i) = i
. . . . . | | | . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . 10. i@0 : |R|
. . . . . | | | . . 11. ¬i@0 = i
. . . . . | | | . . 12. j : |R|
. . . . . | | | . . 13. n : ℕ
. . . . . | | | . . 14. ∀n1:ℕ.
. . . . . | | | . . |   n1<n
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . 15. λx.destination(out(x))^n-k
. . . . . | | | . . |   (λx.destination(out(x))^k(i))
. . . . . | | | . . |   =
. . . . . | | | . . |   j
. . . . . | | | . . 16. ¬n<k
. . . . . | | | . . ⊢  λx.destination(out(x))^n-k(i) = j
. . . . . | | | . . | by NthHypEq -2 THEN Analyze

. . . . . | | | . . |    THEN
. . . . . | | | . . |    Try (Complete Auto)
. . . . . | | | . . |
```

```
. . . . . | | | . . .....subterm .....
. . . . . | | | . . 1. R : Id→𝔹
. . . . . | | | . . 2. in : |R|→IdLnk
. . . . . | | | . . 3. out : |R|→IdLnk
. . . . . | | | . . 4. ∀i:|R|.
. . . . . | | | . . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . |  & source(out(i)) = i
. . . . . | | | . . |    & destination(in(i)) = i
. . . . . | | | . . |    & in(destination(out(i))) = out(i)
. . . . . | | | . . |    & out(source(in(i))) = in(i)
. . . . . | | | . . 5. ∀i,j:|R|.
. . . . . | | | . . |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . 6. i : |R|
. . . . . | | | . . 7. k : ℕ+
. . . . . | | | . . 8. λx.destination(out(x))^k(i) = i
. . . . . | | | . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . 10. i@0 : |R|
. . . . . | | | . . 11. ¬i@0 = i
. . . . . | | | . . 12. j : |R|
. . . . . | | | . . 13. n : ℕ
. . . . . | | | . . 14. ∀n1:ℕ.
. . . . . | | | . . |   n1<n
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . |   ⇒
. . . . . | | | . . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . 15. λx.destination(out(x))^n-k
. . . . . | | | . . |   (λx.destination(out(x))^k(i))
. . . . . | | | . . |   =
. . . . . | | | . . |   j
. . . . . | | | . . 16. ¬n<k
. . . . . | | | . . ⊢  λx.destination(out(x))^n-k(i)
. . . . . | | | . . |  =
. . . . . | | | . . |  λx.destination(out(x))^n-k
. . . . . | | | . . |  (λx.destination(out(x))^k(i))
. . . . . | | | . . | by MoveToConcl 8
. . . . . | | | . . |    THEN
. . . . . | | | . . |    GenConcl ((λx.destination(out(x))) = f)
. . . . . | | | . . |
. . . . . | | | . . |\
. . . . . | | | . . | 1. R : Id→𝔹
. . . . . | | | . . | 2. in : |R|→IdLnk
. . . . . | | | . . | 3. out : |R|→IdLnk
. . . . . | | | . . | 4. ∀i:|R|.
. . . . . | | | . . | |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | |  & source(out(i)) = i
. . . . . | | | . . | |    & destination(in(i)) = i
. . . . . | | | . . | |    & in(destination(out(i))) = out(i)
. . . . . | | | . . | |    & out(source(in(i))) = in(i)
. . . . . | | | . . | 5. ∀i,j:|R|.
. . . . . | | | . . | |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | 6. i : |R|
. . . . . | | | . . | 7. k : ℕ+
. . . . . | | | . . | 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | 9. i@0 : |R|
```

```
. . . . . | | | . . | 10. ¬i@0 = i
. . . . . | | | . . | 11. j : |R|
. . . . . | | | . . | 12. n : ℕ
. . . . . | | | . . | 13. ∀n1:ℕ.
. . . . . | | | . . | |   n1<n
. . . . . | | | . . | |   ⇒
. . . . . | | | . . | |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . | |   ⇒
. . . . . | | | . . | |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | 14. λx.destination(out(x))^n-k
. . . . . | | | . . | |   (λx.destination(out(x))^k(i))
. . . . . | | | . . | |   =
. . . . . | | | . . | |   j
. . . . . | | | . . | 15. ¬n<k
. . . . . | | | . . | 16. f : |R|→|R|
. . . . . | | | . . | 17. (λx.destination(out(x))) = f
. . . . . | | | . . | 18. f^k(i) = i
. . . . . | | | . . | ⊢  f^n-k(i) = f^n-k(f^k(i))
. . . . . | | | . . | | by StrongHypSubst -1 0
. . . . . | | | . . | |    THEN
. . . . . | | | . . | |    Try (Fold ‘member‘ 0)
. . . . . | | | . . | |
. . . . . | | | . . | |\
. . . . . | | | . . | | 1. R : Id→𝔹
. . . . . | | | . . | | 2. in : |R|→IdLnk
. . . . . | | | . . | | 3. out : |R|→IdLnk
. . . . . | | | . . | | 4. ∀i:|R|.
. . . . . | | | . . | |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | |    & source(out(i)) = i
. . . . . | | | . . | |      & destination(in(i)) = i
. . . . . | | | . . | |      & in(destination(out(i))) = out(i)
. . . . . | | | . . | |      & out(source(in(i))) = in(i)
. . . . . | | | . . | | 5. ∀i,j:|R|.
. . . . . | | | . . | |    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | | 6. i : |R|
. . . . . | | | . . | | 7. k : ℕ+
. . . . . | | | . . | | 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | | 9. i@0 : |R|
. . . . . | | | . . | | 10. ¬i@0 = i
. . . . . | | | . . | | 11. j : |R|
. . . . . | | | . . | | 12. n : ℕ
. . . . . | | | . . | | 13. ∀n1:ℕ.
. . . . . | | | . . | |     n1<n
. . . . . | | | . . | |     ⇒
. . . . . | | | . . | |     λx.destination(out(x))^n1(i) = j
. . . . . | | | . . | |     ⇒
. . . . . | | | . . | |     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | | 14. λx.destination(out(x))^n-k
. . . . . | | | . . | |     (λx.destination(out(x))^k(i))
. . . . . | | | . . | |     =
. . . . . | | | . . | |     j
. . . . . | | | . . | | 15. ¬n<k
. . . . . | | | . . | | 16. f : |R|→|R|
. . . . . | | | . . | | 17. (λx.destination(out(x))) = f
. . . . . | | | . . | | 18. f^k(i) = i
```

```
. . . . . | | | . . | | ⊢  f^n-k(i) ∈ Id
. . . . . | | | . . | |   by DoSubsume THEN Analyze 0 THEN Analyze -1
. . . . . | | | . . | |
. . . . . | | | . . | |\
. . . . . | | | . . | | .....subterm .....
. . . . . | | | . . | | 1. R : Id→𝔹
. . . . . | | | . . | | 2. in : |R|→IdLnk
. . . . . | | | . . | | 3. out : |R|→IdLnk
. . . . . | | | . . | | 4. ∀i:|R|.
. . . . . | | | . . | |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | |    & source(out(i)) = i
. . . . . | | | . . | |      & destination(in(i)) = i
. . . . . | | | . . | |      & in(destination(out(i))) = out(i)
. . . . . | | | . . | |      & out(source(in(i))) = in(i)
. . . . . | | | . . | | 5. ∀i,j:|R|.
. . . . . | | | . . | |    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | | 6. i : |R|
. . . . . | | | . . | | 7. k : ℕ+
. . . . . | | | . . | | 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | | 9. i@0 : |R|
. . . . . | | | . . | | 10. ¬i@0 = i
. . . . . | | | . . | | 11. j : |R|
. . . . . | | | . . | | 12. n : ℕ
. . . . . | | | . . | | 13. ∀n1:ℕ.
. . . . . | | | . . | |     n1<n
. . . . . | | | . . | |     ⇒
. . . . . | | | . . | |     λx.destination(out(x))^n1(i) = j
. . . . . | | | . . | |     ⇒
. . . . . | | | . . | |     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | | 14. λx.destination(out(x))^n-k
. . . . . | | | . . | |     (λx.destination(out(x))^k(i))
. . . . . | | | . . | |     =
. . . . . | | | . . | |     j
. . . . . | | | . . | | 15. ¬n<k
. . . . . | | | . . | | 16. f : |R|→|R|
. . . . . | | | . . | | 17. (λx.destination(out(x))) = f
. . . . . | | | . . | | 18. f^k(i) = i
. . . . . | | | . . | | 19. z : Id
. . . . . | | | . . | | 20. z = i
. . . . . | | | . . | | ⊢  f^n-k(i) ∈ Id
. . . . . | | | . . | |   by DoSubsume THEN Analyze 0 THEN Analyze -1
. . . . . | | | . . | |
. . . . . | | | . . | .\
. . . . . | | | . . | . .....subterm .....
. . . . . | | | . . | . 1. R : Id→𝔹
. . . . . | | | . . | . 2. in : |R|→IdLnk
. . . . . | | | . . | . 3. out : |R|→IdLnk
. . . . . | | | . . | . 4. ∀i:|R|.
. . . . . | | | . . | . |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | . |  & source(out(i)) = i
. . . . . | | | . . | . |    & destination(in(i)) = i
. . . . . | | | . . | . |    & in(destination(out(i))) = out(i)
. . . . . | | | . . | . |    & out(source(in(i))) = in(i)
. . . . . | | | . . | . 5. ∀i,j:|R|.
. . . . . | | | . . | . |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
```

```
. . . . . | | | . . . | . 6. i : |R|
. . . . . | | | . . . | . 7. k : ℕ+
. . . . . | | | . . . | . 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . . | . 9. i@0 : |R|
. . . . . | | | . . . | . 10. ¬i@0 = i
. . . . . | | | . . . | . 11. j : |R|
. . . . . | | | . . . | . 12. n : ℕ
. . . . . | | | . . . | . 13. ∀n1:ℕ.
. . . . . | | | . . . | . |   n1<n
. . . . . | | | . . . | . |   ⇒
. . . . . | | | . . . | . |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . . | . |   ⇒
. . . . . | | | . . . | . |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . . | . 14. λx.destination(out(x))^n-k
. . . . . | | | . . . | . |   (λx.destination(out(x))^k(i))
. . . . . | | | . . . | . |   =
. . . . . | | | . . . | . |   j
. . . . . | | | . . . | . 15. ¬n<k
. . . . . | | | . . . | . 16. f : |R|→|R|
. . . . . | | | . . . | . 17. (λx.destination(out(x))) = f
. . . . . | | | . . . | . 18. f^k(i) = i
. . . . . | | | . . . | . 19. z : Id
. . . . . | | | . . . | . 20. z = i
. . . . . | | | . . . | . ⊢  f^n-k(z) ∈ Id  by Assert (z ∈ |R|)
. . . . . | | | . . . | . |
. . . . . | | | . . . | . |\
. . . . . | | | . . . | . | .....assertion .....
. . . . . | | | . . . | . | 1. R : Id→𝔹
. . . . . | | | . . . | . | 2. in : |R|→IdLnk
. . . . . | | | . . . | . | 3. out : |R|→IdLnk
. . . . . | | | . . . | . | 4. ∀i:|R|.
. . . . . | | | . . . | . | |  R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . . | . | |  & source(out(i)) = i
. . . . . | | | . . . | . | |    & destination(in(i)) = i
. . . . . | | | . . . | . | |    & in(destination(out(i))) = out(i)
. . . . . | | | . . . | . | |    & out(source(in(i))) = in(i)
. . . . . | | | . . . | . | 5. ∀i,j:|R|.
. . . . . | | | . . . | . | |  ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . . | . | 6. i : |R|
. . . . . | | | . . . | . | 7. k : ℕ+
. . . . . | | | . . . | . | 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . . | . | 9. i@0 : |R|
. . . . . | | | . . . | . | 10. ¬i@0 = i
. . . . . | | | . . . | . | 11. j : |R|
. . . . . | | | . . . | . | 12. n : ℕ
. . . . . | | | . . . | . | 13. ∀n1:ℕ.
. . . . . | | | . . . | . | |   n1<n
. . . . . | | | . . . | . | |   ⇒
. . . . . | | | . . . | . | |   λx.destination(out(x))^n1(i) = j
. . . . . | | | . . . | . | |   ⇒
. . . . . | | | . . . | . | |   (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . . | . | 14. λx.destination(out(x))^n-k
. . . . . | | | . . . | . | |   (λx.destination(out(x))^k(i))
. . . . . | | | . . . | . | |   =
. . . . . | | | . . . | . | |   j
```

```
. . . . . | | | . . | . | 15. ¬n<k
. . . . . | | | . . | . | 16. f : |R|→|R|
. . . . . | | | . . | . | 17. (λx.destination(out(x))) = f
. . . . . | | | . . | . | 18. f^k(i) = i
. . . . . | | | . . | . | 19. z : Id
. . . . . | | | . . | . | 20. z = i
. . . . . | | | . . | . | ⊢  z ∈ |R|  by DVar ‘i’ THEN Analyze
. . . . . | | | . . | . | |
. . . . . | | | . . | . | .....set predicate .....
. . . . . | | | . . | . | 1. R : Id→𝔹
. . . . . | | | . . | . | 2. in : |R|→IdLnk
. . . . . | | | . . | . | 3. out : |R|→IdLnk
. . . . . | | | . . | . | 4. ∀i:|R|.
. . . . . | | | . . | . |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | . |    & source(out(i)) = i
. . . . . | | | . . | . |      & destination(in(i)) = i
. . . . . | | | . . | . |      & in(destination(out(i))) = out(i)
. . . . . | | | . . | . |      & out(source(in(i))) = in(i)
. . . . . | | | . . | . | 5. ∀i,j:|R|.
. . . . . | | | . . | . |    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | . | 6. i : Id
. . . . . | | | . . | . | 7. R(i)
. . . . . | | | . . | . | 8. k : ℕ+
. . . . . | | | . . | . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | . | 10. i@0 : |R|
. . . . . | | | . . | . | 11. ¬i@0 = i
. . . . . | | | . . | . | 12. j : |R|
. . . . . | | | . . | . | 13. n : ℕ
. . . . . | | | . . | . | 14. ∀n1:ℕ.
. . . . . | | | . . | . |     n1<n
. . . . . | | | . . | . |     ⇒
. . . . . | | | . . | . |     λx.destination(out(x))^n1(i) = j
. . . . . | | | . . | . |     ⇒
. . . . . | | | . . | . |     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | . | 15. λx.destination(out(x))^n-k
. . . . . | | | . . | . |     (λx.destination(out(x))^k(i))
. . . . . | | | . . | . |     =
. . . . . | | | . . | . |     j
. . . . . | | | . . | . | 16. ¬n<k
. . . . . | | | . . | . | 17. f : |R|→|R|
. . . . . | | | . . | . | 18. (λx.destination(out(x))) = f
. . . . . | | | . . | . | 19. f^k(i) = i
. . . . . | | | . . | . | 20. z : Id
. . . . . | | | . . | . | 21. z = i
. . . . . | | | . . | . | ⊢  R(z)  by HypSubst -1 0
. . . . . | | | . . | . |
. . . . . | | | . . | . .\
. . . . . | | | . . | . . 1. R : Id→𝔹
. . . . . | | | . . | . . 2. in : |R|→IdLnk
. . . . . | | | . . | . . 3. out : |R|→IdLnk
. . . . . | | | . . | . . 4. ∀i:|R|.
. . . . . | | | . . | . .    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . | . .    & source(out(i)) = i
. . . . . | | | . . | . .      & destination(in(i)) = i
. . . . . | | | . . | . .      & in(destination(out(i))) = out(i)
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. . . . . | | | . . | . .      & out(source(in(i))) = in(i)
. . . . . | | | . . | . . 5. ∀i,j:|R|.
. . . . . | | | . . | . .    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | . . 6. i : |R|
. . . . . | | | . . | . . 7. k : ℕ+
. . . . . | | | . . | . . 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | . . 9. i@0 : |R|
. . . . . | | | . . | . . 10. ¬i@0 = i
. . . . . | | | . . | . . 11. j : |R|
. . . . . | | | . . | . . 12. n : ℕ
. . . . . | | | . . | . . 13. ∀n1:ℕ.
. . . . . | | | . . | . .     n1<n
. . . . . | | | . . | . .     ⇒
. . . . . | | | . . | . .     λx.destination(out(x))^n1(i) = j
. . . . . | | | . . | . .     ⇒
. . . . . | | | . . | . .     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | . . 14. λx.destination(out(x))^n-k
. . . . . | | | . . | . .     (λx.destination(out(x))^k(i))
. . . . . | | | . . | . .     =
. . . . . | | | . . | . .     j
. . . . . | | | . . | . . 15. ¬n<k
. . . . . | | | . . | . . 16. f : |R|→|R|
. . . . . | | | . . | . . 17. (λx.destination(out(x))) = f
. . . . . | | | . . | . . 18. f^k(i) = i
. . . . . | | | . . | . . 19. z : Id
. . . . . | | | . . | . . 20. z = i
. . . . . | | | . . | . . 21. z ∈ |R|
. . . . . | | | . . | . . ⊢  f^n-k(z) ∈ Id
. . . . . | | | . . | . .   by DoSubsume THEN Analyze 0
. . . . . | | | . . | . .      THEN
. . . . . | | | . . | . .      Analyze -1
. . . . . | | | . . |
. . . . . | | | . . |\
. . . . . | | | . . | .....subterm .....
. . . . . | | | . . | 1. R : Id→𝔹
. . . . . | | | . . | 2. in : |R|→IdLnk
. . . . . | | | . . | 3. out : |R|→IdLnk
. . . . . | | | . . | 4. ∀i:|R|.
. . . . . | | | . . |    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . |    & source(out(i)) = i
. . . . . | | | . . |      & destination(in(i)) = i
. . . . . | | | . . |      & in(destination(out(i))) = out(i)
. . . . . | | | . . |      & out(source(in(i))) = in(i)
. . . . . | | | . . | 5. ∀i,j:|R|.
. . . . . | | | . . |    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . | 6. i : |R|
. . . . . | | | . . | 7. k : ℕ+
. . . . . | | | . . | 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . | 9. i@0 : |R|
. . . . . | | | . . | 10. ¬i@0 = i
. . . . . | | | . . | 11. j : |R|
. . . . . | | | . . | 12. n : ℕ
. . . . . | | | . . | 13. ∀n1:ℕ.
. . . . . | | | . . |     n1<n
. . . . . | | | . . |     ⇒
```

```
. . . . . | | | . . |      λx.destination(out(x))^n1(i) = j
. . . . . | | | . . |      ⇒
. . . . . | | | . . |      (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . | 14. λx.destination(out(x))^n-k
. . . . . | | | . . |      (λx.destination(out(x))^k(i))
. . . . . | | | . . |      =
. . . . . | | | . . |      j
. . . . . | | | . . | 15. ¬n<k
. . . . . | | | . . | 16. f : |R|→|R|
. . . . . | | | . . | 17. (λx.destination(out(x))) = f
. . . . . | | | . . | ⊢  f^k(i) ∈ Id
. . . . . | | | . . |   by DoSubsume THEN Analyze 0 THEN Analyze -1
. . . . . | | | . . |
. . . . . | | | . . .\
. . . . . | | | . . . .....subterm .....
. . . . . | | | . . . 1. R : Id→𝔹
. . . . . | | | . . . 2. in : |R|→IdLnk
. . . . . | | | . . . 3. out : |R|→IdLnk
. . . . . | | | . . . 4. ∀i:|R|.
. . . . . | | | . . .    R(source(in(i))) & R(destination(out(i)))
. . . . . | | | . . .    & source(out(i)) = i
. . . . . | | | . . .      & destination(in(i)) = i
. . . . . | | | . . .      & in(destination(out(i))) = out(i)
. . . . . | | | . . .      & out(source(in(i))) = in(i)
. . . . . | | | . . . 5. ∀i,j:|R|.
. . . . . | | | . . .    ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | | . . . 6. i : |R|
. . . . . | | | . . . 7. k : ℕ+
. . . . . | | | . . . 8. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | | . . . 9. i@0 : |R|
. . . . . | | | . . . 10. ¬i@0 = i
. . . . . | | | . . . 11. j : |R|
. . . . . | | | . . . 12. n : ℕ
. . . . . | | | . . . 13. ∀n1:ℕ.
. . . . . | | | . . .     n1<n
. . . . . | | | . . .     ⇒
. . . . . | | | . . .     λx.destination(out(x))^n1(i) = j
. . . . . | | | . . .     ⇒
. . . . . | | | . . .     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | | . . . 14. λx.destination(out(x))^n-k
. . . . . | | | . . .     (λx.destination(out(x))^k(i))
. . . . . | | | . . .     =
. . . . . | | | . . .     j
. . . . . | | | . . . 15. ¬n<k
. . . . . | | | . . . 16. f : |R|→|R|
. . . . . | | | . . . 17. (λx.destination(out(x))) = f
. . . . . | | | . . . ⊢  i ∈ Id  by DVar ‘i’
. . . . . | | |
. . . . . | | .\
. . . . . | | . .....wf .....
. . . . . | | . 1. R : Id→𝔹
. . . . . | | . 2. in : |R|→IdLnk
. . . . . | | . 3. out : |R|→IdLnk
. . . . . | | . 4. ∀i:|R|.
. . . . . | | .    R(source(in(i))) & R(destination(out(i)))
```

```
. . . . . | | .    & source(out(i)) = i
. . . . . | | .      & destination(in(i)) = i
. . . . . | | .      & in(destination(out(i))) = out(i)
. . . . . | | .      & out(source(in(i))) = in(i)
. . . . . | | . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | | . 6. i : |R|
. . . . . | | . 7. k : ℕ+
. . . . . | | . 8. λx.destination(out(x))^k(i) = i
. . . . . | | . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | | . 10. i@0 : |R|
. . . . . | | . 11. ¬i@0 = i
. . . . . | | . 12. j : |R|
. . . . . | | . 13. n : ℕ
. . . . . | | . 14. ∀n1:ℕ.
. . . . . | | .     n1<n
. . . . . | | .     ⇒
. . . . . | | .     λx.destination(out(x))^n1(i) = j
. . . . . | | .     ⇒
. . . . . | | .     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | | . ⊢  (λx.destination(out(x))^n(i) = j) ∈ Prop
. . . . . | | .   by GenConcl ((λx.destination(out(x))) = f)
. . . . . | | .      THEN
. . . . . | | .      DoSubsume
. . . . . | | .      THEN
. . . . . | | .      Analyze 0
. . . . . | | .      THEN
. . . . . | | .      Analyze -1
. . . . . | |
. . . . . | .\
. . . . . | . 1. R : Id→𝔹
. . . . . | . 2. in : |R|→IdLnk
. . . . . | . 3. out : |R|→IdLnk
. . . . . | . 4. ∀i:|R|.
. . . . . | .    R(source(in(i))) & R(destination(out(i)))
. . . . . | .    & source(out(i)) = i
. . . . . | .      & destination(in(i)) = i
. . . . . | .      & in(destination(out(i))) = out(i)
. . . . . | .      & out(source(in(i))) = in(i)
. . . . . | . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . | . 6. i : |R|
. . . . . | . 7. k : ℕ+
. . . . . | . 8. λx.destination(out(x))^k(i) = i
. . . . . | . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . | . 10. i@0 : |R|
. . . . . | . 11. ¬i@0 = i
. . . . . | . 12. j : |R|
. . . . . | . 13. ∀n:ℕ.
. . . . . | .     λx.destination(out(x))^n(i) = j
. . . . . | .     ⇒
. . . . . | .     (∃m:ℕk. λx.destination(out(x))^m(i) = j)
. . . . . | . ⊢  ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . | .   by InstHyp [i;j] 5 THEN ExRepD THEN InstHyp [k1] -3
. . . . . |
. . . . . .\
. . . . . . 1. R : Id→𝔹
```

```
. . . . . . . 2. in : |R|→IdLnk
. . . . . . . 3. out : |R|→IdLnk
. . . . . . . 4. ∀i:|R|.
. . . . . . . |  R(source(in(i))) & R(destination(out(i)))
. . . . . . . |  & source(out(i)) = i
. . . . . . . |    & destination(in(i)) = i
. . . . . . . |    & in(destination(out(i))) = out(i)
. . . . . . . |    & out(source(in(i))) = in(i)
. . . . . . . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . . . 6. i : |R|
. . . . . . . 7. k : ℕ+
. . . . . . . 8. λx.destination(out(x))^k(i) = i
. . . . . . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . . . 10. i@0 : |R|
. . . . . . . 11. ¬i@0 = i
. . . . . . . 12. ∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . . . ⊢  ∃y:ℕk.
. . . . . . . |  (y ∈ upto(k)) & i@0 = λx.destination(out(x))^y(i)
. . . . . . . | by InstHyp [i@0] -1 THEN ParallelOp -1
. . . . . . . |
. . . . . . . |\
. . . . . . . | 1. R : Id→𝔹
. . . . . . . | 2. in : |R|→IdLnk
. . . . . . . | 3. out : |R|→IdLnk
. . . . . . . | 4. ∀i:|R|.
. . . . . . . |    R(source(in(i))) & R(destination(out(i)))
. . . . . . . |    & source(out(i)) = i
. . . . . . . |      & destination(in(i)) = i
. . . . . . . |      & in(destination(out(i))) = out(i)
. . . . . . . |      & out(source(in(i))) = in(i)
. . . . . . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . . . | 6. i : |R|
. . . . . . . | 7. k : ℕ+
. . . . . . . | 8. λx.destination(out(x))^k(i) = i
. . . . . . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . . . | 10. i@0 : |R|
. . . . . . . | 11. ¬i@0 = i
. . . . . . . | 12. ∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . . . | 13. n : ℕk
. . . . . . . | 14. λx.destination(out(x))^n(i) = i@0
. . . . . . . | ⊢  (n ∈ upto(k))
. . . . . . . |   by BackThru
. . . . . . . |      Thm* ∀n,i:ℕ. i<n ⇒ (i ∈ upto(n))
. . . . . . . |      {member_upto2}
. . . . . . . |
. . . . . . . |\
. . . . . . . | 1. R : Id→𝔹
. . . . . . . | 2. in : |R|→IdLnk
. . . . . . . | 3. out : |R|→IdLnk
. . . . . . . | 4. ∀i:|R|.
. . . . . . . |    R(source(in(i))) & R(destination(out(i)))
. . . . . . . |    & source(out(i)) = i
. . . . . . . |      & destination(in(i)) = i
. . . . . . . |      & in(destination(out(i))) = out(i)
. . . . . . . |      & out(source(in(i))) = in(i)
```

```
. . . . . . . | 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . . . | 6. i : |R|

. . . . . . . | 7. k : ℕ+
. . . . . . . | 8. λx.destination(out(x))^k(i) = i
. . . . . . . | 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . . . | 10. i@0 : |R|
. . . . . . . | 11. ¬i@0 = i
. . . . . . . | 12. ∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . . . | 13. n : ℕk
. . . . . . . | 14. λx.destination(out(x))^n(i) = i@0
. . . . . . . | ⊢  i@0 = λx.destination(out(x))^n(i)
. . . . . . . |   by StrongHypSubst -1 0 THEN Analyze THEN DVar ‘i@0’
. . . . . . . |      THEN
. . . . . . . |      HypSubst -1 0
. . . . . . . |      THEN
. . . . . . . |      Unhide
. . . . . . . |
. . . . . . . .\
. . . . . . . . .....subterm .....
. . . . . . . . 1. R : Id→𝔹
. . . . . . . . 2. in : |R|→IdLnk
. . . . . . . . 3. out : |R|→IdLnk
. . . . . . . . 4. ∀i:|R|.
. . . . . . .      R(source(in(i))) & R(destination(out(i)))
. . . . . . .      & source(out(i)) = i
. . . . . . .        & destination(in(i)) = i
. . . . . . .        & in(destination(out(i))) = out(i)
. . . . . . .        & out(source(in(i))) = in(i)
. . . . . . . . 5. ∀i,j:|R|. ∃k:ℕ+. λx.destination(out(x))^k(i) = j
. . . . . . . . 6. i : |R|
. . . . . . . . 7. k : ℕ+
. . . . . . . . 8. λx.destination(out(x))^k(i) = i
. . . . . . . . 9. (λx.destination(out(x))) ∈ |R|→|R|
. . . . . . . . 10. i@0 : |R|
. . . . . . . . 11. ¬i@0 = i
. . . . . . . . 12. ∀j:|R|. ∃n:ℕk. λx.destination(out(x))^n(i) = j
. . . . . . . . 13. n : ℕk
. . . . . . . . 14. λx.destination(out(x))^n(i) = i@0
. . . . . . . . 15. y : ℕk
. . . . . . . . 16. (y ∈ upto(k))
. . . . . . . . ⊢  λx.destination(out(x))^y(i) ∈ |R|
. . . . . . .     by GenConcl ((λx.destination(out(x))) = f)
END
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