```
rel_star_finite  THM

⊢  ∀n:ℕ, R:(ℕn→ℕn→Prop), x,y:ℕn. (x (R^*) y) ⟺ (∃k:ℕ(n+1). x R^k y)
| by Auto
|
|\
| 1. n : ℕ
| 2. R : ℕn→ℕn→Prop
| 3. x : ℕn
| 4. y : ℕn
| 5. x (R^*) y
| ⊢  ∃k:ℕ(n+1). x R^k y
| | by Unfold `rel_star` (-1) THEN Reduce (-1) THEN ExRepD
| |
| 1. n : ℕ
| 2. R : ℕn→ℕn→Prop
| 3. x : ℕn
| 4. y : ℕn
| 5. n@0 : ℕ
| 6. x R^n@0 y
| ⊢  ∃k:ℕ(n+1). x R^k y
| | by ...
| |
| 1. n : ℕ
| 2. R : ℕn→ℕn→Prop
| 3. x : ℕn
| 4. y : ℕn
| 5. n@0 : ℕ
| 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| 7. x R^n@0 y
| ⊢  ∃k:ℕ(n+1). x R^k y  by Decide n@0≤n THENA Auto
| |
| |\
| | 1. n : ℕ
| | 2. R : ℕn→ℕn→Prop
| | 3. x : ℕn
| | 4. y : ℕn
| | 5. n@0 : ℕ
| | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| | 7. x R^n@0 y
| | 8. n@0≤n
| | ⊢  ∃k:ℕ(n+1). x R^k y  by InstConcl [n@0] THEN Auto'
| |
| .\
| . 1. n : ℕ
| . 2. R : ℕn→ℕn→Prop
| . 3. x : ℕn
| . 4. y : ℕn
| . 5. n@0 : ℕ
| . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . 7. x R^n@0 y
| . 8. ¬n@0≤n
| . ⊢  ∃k:ℕ(n+1). x R^k y
| . | by RWO "rel_exp_list" (-2) THENA Auto THEN ExRepD
| . |
```

```
| . 1. n : ℕ
| . 2. R : ℕn→ℕn→Prop
| . 3. x : ℕn
| . 4. y : ℕn
| . 5. n@0 : ℕ
| . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . 7. L : ℕn List
| . 8. ||L|| = n@0+1
| . 9. L[0] = x
| . 10. last(L) = y
| . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . 12. ¬n@0≤n
| . ⊢  ∃k:ℕ(n+1). x R^k y  by Assert ∃i,j:ℕ(n@0+1). i<j & L[i] = L[j]
| . |
| . |\
| . | .....assertion .....
| . | 1. n : ℕ
| . | 2. R : ℕn→ℕn→Prop
| . | 3. x : ℕn
| . | 4. y : ℕn
| . | 5. n@0 : ℕ
| . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | 7. L : ℕn List
| . | 8. ||L|| = n@0+1
| . | 9. L[0] = x
| . | 10. last(L) = y
| . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | 12. ¬n@0≤n
| . | ⊢  ∃i,j:ℕ(n@0+1). i<j & L[i] = L[j]  by SupposeNot
| . | |
| . | 1. n : ℕ
| . | 2. R : ℕn→ℕn→Prop
| . | 3. x : ℕn
| . | 4. y : ℕn
| . | 5. n@0 : ℕ
| . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | 7. L : ℕn List
| . | 8. ||L|| = n@0+1
| . | 9. L[0] = x
| . | 10. last(L) = y
| . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | 12. ¬n@0≤n
| . | 13. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | ⊢  ∃i,j:ℕ(n@0+1). i<j & L[i] = L[j]
| . | | by D (-2) THEN BackThruLemma ‘injection_le‘ THEN Auto
| . | |
| . | 1. n : ℕ
| . | 2. R : ℕn→ℕn→Prop
| . | 3. x : ℕn
| . | 4. y : ℕn
| . | 5. n@0 : ℕ
| . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | 7. L : ℕn List
| . | 8. ||L|| = n@0+1
| . | 9. L[0] = x
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| . | 10. last(L) = y
| . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | ⊢  ∃f:(ℕn@0→ℕn). Inj(ℕn@0; ℕn; f)
| . | | by InstConcl [λi.L[(i+1)]] THEN Auto
| . | |
| . | 1. n : ℕ
| . | 2. R : ℕn→ℕn→Prop
| . | 3. x : ℕn
| . | 4. y : ℕn
| . | 5. n@0 : ℕ
| . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | 7. L : ℕn List
| . | 8. ||L|| = n@0+1
| . | 9. L[0] = x
| . | 10. last(L) = y
| . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | ⊢  Inj(ℕn@0; ℕn; λi.L[(i+1)])
| . | | by Unfold ‘inject‘ 0 THEN Reduce 0 THEN Auto
| . | |
| . | 1. n : ℕ
| . | 2. R : ℕn→ℕn→Prop
| . | 3. x : ℕn
| . | 4. y : ℕn
| . | 5. n@0 : ℕ
| . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | 7. L : ℕn List
| . | 8. ||L|| = n@0+1
| . | 9. L[0] = x
| . | 10. last(L) = y
| . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | 13. a1 : ℕn@0
| . | 14. a2 : ℕn@0
| . | 15. L[(a1+1)] = L[(a2+1)]
| . | ⊢  a1 = a2  by Decide a1<a2 THENA Auto
| . | |
| . | |\
| . | | 1. n : ℕ
| . | | 2. R : ℕn→ℕn→Prop
| . | | 3. x : ℕn
| . | | 4. y : ℕn
| . | | 5. n@0 : ℕ
| . | | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | | 7. L : ℕn List
| . | | 8. ||L|| = n@0+1
| . | | 9. L[0] = x
| . | | 10. last(L) = y
| . | | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | | 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | | 13. a1 : ℕn@0
| . | | 14. a2 : ℕn@0
| . | | 15. L[(a1+1)] = L[(a2+1)]
| . | | 16. a1<a2
```

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| . | | ⊢  a1 = a2  by D (-5) THEN InstConcl [a1+1;a2+1] THEN Auto'
| . | |
| . | .\
| . | . 1. n : ℕ
| . | . 2. R : ℕn→ℕn→Prop
| . | . 3. x : ℕn
| . | . 4. y : ℕn
| . | . 5. n@0 : ℕ
| . | . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | . 7. L : ℕn List
| . | . 8. ||L|| = n@0+1
| . | . 9. L[0] = x
| . | . 10. last(L) = y
| . | . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | . 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | . 13. a1 : ℕn@0
| . | . 14. a2 : ℕn@0
| . | . 15. L[(a1+1)] = L[(a2+1)]
| . | . 16. ¬a1<a2
| . | . ⊢  a1 = a2  by Decide a2<a1 THENA Auto
| . | . |
| . | . |\
| . | . | 1. n : ℕ
| . | . | 2. R : ℕn→ℕn→Prop
| . | . | 3. x : ℕn
| . | . | 4. y : ℕn
| . | . | 5. n@0 : ℕ
| . | . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | . | 7. L : ℕn List
| . | . | 8. ||L|| = n@0+1
| . | . | 9. L[0] = x
| . | . | 10. last(L) = y
| . | . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | . | 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | . | 13. a1 : ℕn@0
| . | . | 14. a2 : ℕn@0
| . | . | 15. L[(a1+1)] = L[(a2+1)]
| . | . | 16. ¬a1<a2
| . | . | 17. a2<a1
| . | . | ⊢  a1 = a2  by D (-6) THEN InstConcl [a2+1;a1+1] THEN Auto'
| . | . |
| . | . .\
| . | . . 1. n : ℕ
| . | . . 2. R : ℕn→ℕn→Prop
| . | . . 3. x : ℕn
| . | . . 4. y : ℕn
| . | . . 5. n@0 : ℕ
| . | . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . | . . 7. L : ℕn List
| . | . . 8. ||L|| = n@0+1
| . | . . 9. L[0] = x
| . | . . 10. last(L) = y
| . | . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . | . . 12. ¬(∃i,j:ℕ(n@0+1). i<j & L[i] = L[j])
| . | . . 13. a1 : ℕn@0
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| . | . . 14. a2 : ℕn@0
| . | . . 15. L[(a1+1)] = L[(a2+1)]
| . | . . 16. ¬a1<a2
| . | . . 17. ¬a2<a1
| . | . . ⊢  a1 = a2  by Auto
| . |
| . .\
| . . 1. n : ℕ
| . . 2. R : ℕn→ℕn→Prop
| . . 3. x : ℕn
| . . 4. y : ℕn
| . . 5. n@0 : ℕ
| . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . 7. L : ℕn List
| . . 8. ||L|| = n@0+1
| . . 9. L[0] = x
| . . 10. last(L) = y
| . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . 12. ¬n@0≤n
| . . 13. ∃i,j:ℕ(n@0+1). i<j & L[i] = L[j]
| . . ⊢  ∃k:ℕ(n+1). x R^k y  by ExRepD
| . . |
| . . 1. n : ℕ
| . . 2. R : ℕn→ℕn→Prop
| . . 3. x : ℕn
| . . 4. y : ℕn
| . . 5. n@0 : ℕ
| . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . 7. L : ℕn List
| . . 8. ||L|| = n@0+1
| . . 9. L[0] = x
| . . 10. last(L) = y
| . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . 12. ¬n@0≤n
| . . 13. i : ℕ(n@0+1)
| . . 14. j : ℕ(n@0+1)
| . . 15. i<j
| . . 16. L[i] = L[j]
| . . ⊢  ∃k:ℕ(n+1). x R^k y  by InstHyp [n@0-(j-i)] 6 THEN Auto'
| . . |
| . . .....antecedent .....
| . . 1. n : ℕ
| . . 2. R : ℕn→ℕn→Prop
| . . 3. x : ℕn
| . . 4. y : ℕn
| . . 5. n@0 : ℕ
| . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . 7. L : ℕn List
| . . 8. ||L|| = n@0+1
| . . 9. L[0] = x
| . . 10. last(L) = y
| . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . 12. ¬n@0≤n
| . . 13. i : ℕ(n@0+1)
| . . 14. j : ℕ(n@0+1)
```

```
| . . 15. i<j
| . . 16. L[i] = L[j]
| . . ⊢  x R^n@0-(j-i) y  by RWO "rel_exp_list" 0 THEN Auto
| . . |
| . . 1. n : ℕ
| . . 2. R : ℕn→ℕn→Prop
| . . 3. x : ℕn
| . . 4. y : ℕn
| . . 5. n@0 : ℕ
| . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . 7. L : ℕn List
| . . 8. ||L|| = n@0+1
| . . 9. L[0] = x
| . . 10. last(L) = y
| . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . 12. ¬n@0≤n
| . . 13. i : ℕ(n@0+1)
| . . 14. j : ℕ(n@0+1)
| . . 15. i<j
| . . 16. L[i] = L[j]
| . . ⊢  ∃L:ℕn List.
| . . |  ||L|| = n@0-(j-i)+1
| . . |  & L[0] = x & last(L) = y & (∀i:ℕ(n@0-(j-i)). L[i] R L[(i+1)])
| . . | by InstConcl [firstn(i+1;L) @ nth_tl(j+1;L)
| . . |    ] THENA (Auto' THEN NotNullByLength)
| . . |
| . . 1. n : ℕ
| . . 2. R : ℕn→ℕn→Prop
| . . 3. x : ℕn
| . . 4. y : ℕn
| . . 5. n@0 : ℕ
| . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . 7. L : ℕn List
| . . 8. ||L|| = n@0+1
| . . 9. L[0] = x
| . . 10. last(L) = y
| . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . 12. ¬n@0≤n
| . . 13. i : ℕ(n@0+1)
| . . 14. j : ℕ(n@0+1)
| . . 15. i<j
| . . 16. L[i] = L[j]
| . . ⊢  ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . |  & (firstn(i+1;L) @ nth_tl(j+1;L))[0] = x
| . . |    & last(firstn(i+1;L) @ nth_tl(j+1;L)) = y
| . . |    & (∀i@0:ℕ(n@0-(j-i)).
| . . |       (firstn(i+1;L) @ nth_tl(j+1;L))[i@0]
| . . |       R (firstn(i+1;L) @ nth_tl(j+1;L))[(i@0+1)])
| . . | by AssertSubterm [1]
| . . |
| . . |\
| . . | .....assertion .....
| . . | 1. n : ℕ
| . . | 2. R : ℕn→ℕn→Prop
| . . | 3. x : ℕn
```

```
| . . | 4. y : ℕn
| . . | 5. n@0 : ℕ
| . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . | 7. L : ℕn List
| . . | 8. ||L|| = n@0+1
| . . | 9. L[0] = x
| . . | 10. last(L) = y
| . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . | 12. ¬n@0≤n
| . . | 13. i : ℕ(n@0+1)
| . . | 14. j : ℕ(n@0+1)
| . . | 15. i<j
| . . | 16. L[i] = L[j]
| . . | ⊢  ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . |   by ...
| . . |
| . . .\
| . . . 1. n : ℕ
| . . . 2. R : ℕn→ℕn→Prop
| . . . 3. x : ℕn
| . . . 4. y : ℕn
| . . . 5. n@0 : ℕ
| . . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . 7. L : ℕn List
| . . . 8. ||L|| = n@0+1
| . . . 9. L[0] = x
| . . . 10. last(L) = y
| . . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . 12. ¬n@0≤n
| . . . 13. i : ℕ(n@0+1)
| . . . 14. j : ℕ(n@0+1)
| . . . 15. i<j
| . . . 16. L[i] = L[j]
| . . . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . ⊢  ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . |  & (firstn(i+1;L) @ nth_tl(j+1;L))[0] = x
| . . . |    & last(firstn(i+1;L) @ nth_tl(j+1;L)) = y
| . . . |    & (∀i@0:ℕ(n@0-(j-i)).
| . . . |       (firstn(i+1;L) @ nth_tl(j+1;L))[i@0]
| . . . |       R (firstn(i+1;L) @ nth_tl(j+1;L))[(i@0+1)])
| . . . | by SimpConcl
| . . . |
| . . . 1. n : ℕ
| . . . 2. R : ℕn→ℕn→Prop
| . . . 3. x : ℕn
| . . . 4. y : ℕn
| . . . 5. n@0 : ℕ
| . . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . 7. L : ℕn List
| . . . 8. ||L|| = n@0+1
| . . . 9. L[0] = x
| . . . 10. last(L) = y
| . . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . 12. ¬n@0≤n
| . . . 13. i : ℕ(n@0+1)
```

```
| . . . 14. j : ℕ(n@0+1)
| . . . 15. i<j
| . . . 16. L[i] = L[j]
| . . . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[0] = x
| . . . |  & last(firstn(i+1;L) @ nth_tl(j+1;L)) = y
| . . . |  & (∀i@0:ℕ(n@0-(j-i)).
| . . . |     (firstn(i+1;L) @ nth_tl(j+1;L))[i@0]
| . . . |     R (firstn(i+1;L) @ nth_tl(j+1;L))[(i@0+1)])  by Auto
| . . . |
| . . . |\
| . . . | 1. n : ℕ
| . . . | 2. R : ℕn→ℕn→Prop
| . . . | 3. x : ℕn
| . . . | 4. y : ℕn
| . . . | 5. n@0 : ℕ
| . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | 7. L : ℕn List
| . . . | 8. ||L|| = n@0+1
| . . . | 9. L[0] = x
| . . . | 10. last(L) = y
| . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | 12. ¬n@0≤n
| . . . | 13. i : ℕ(n@0+1)
| . . . | 14. j : ℕ(n@0+1)
| . . . | 15. i<j
| . . . | 16. L[i] = L[j]
| . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[0] = x
| . . . |   by ...
| . . . |
| . . . |\
| . . . | 1. n : ℕ
| . . . | 2. R : ℕn→ℕn→Prop
| . . . | 3. x : ℕn
| . . . | 4. y : ℕn
| . . . | 5. n@0 : ℕ
| . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | 7. L : ℕn List
| . . . | 8. ||L|| = n@0+1
| . . . | 9. L[0] = x
| . . . | 10. last(L) = y
| . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | 12. ¬n@0≤n
| . . . | 13. i : ℕ(n@0+1)
| . . . | 14. j : ℕ(n@0+1)
| . . . | 15. i<j
| . . . | 16. L[i] = L[j]
| . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | ⊢  last(firstn(i+1;L) @ nth_tl(j+1;L)) = y
| . . . | | by WeakSubstFor y 0 THENA (Auto' THEN NotNullByLength)
| . . . | |
| . . . | |
| . . . | |
| . . . | 1. n : ℕ
```

```
| . . . | 2. R : ℕn→ℕn→Prop
| . . . | 3. x : ℕn
| . . . | 4. y : ℕn
| . . . | 5. n@0 : ℕ
| . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | 7. L : ℕn List
| . . . | 8. ||L|| = n@0+1
| . . . | 9. L[0] = x
| . . . | 10. last(L) = y
| . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | 12. ¬n@0≤n
| . . . | 13. i : ℕ(n@0+1)
| . . . | 14. j : ℕ(n@0+1)
| . . . | 15. i<j
| . . . | 16. L[i] = L[j]
| . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | ⊢  last(firstn(i+1;L) @ nth_tl(j+1;L)) = last(L)
| . . . | | by ...
| . . . | |
| . . . | 1. n : ℕ
| . . . | 2. R : ℕn→ℕn→Prop
| . . . | 3. x : ℕn
| . . . | 4. y : ℕn
| . . . | 5. n@0 : ℕ
| . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | 7. L : ℕn List
| . . . | 8. ||L|| = n@0+1
| . . . | 9. L[0] = x
| . . . | 10. last(L) = y
| . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | 12. ¬n@0≤n
| . . . | 13. i : ℕ(n@0+1)
| . . . | 14. j : ℕ(n@0+1)
| . . . | 15. i<j
| . . . | 16. L[i] = L[j]
| . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[(n@0-(j-i)+1-1)]
| . . . | |  =
| . . . | |  L[(||L||-1)]
| . . . | | by InstLemma ‘length_firstn‘ [ℕn;L;i+1] THENA Auto’
| . . . |
| . . . | |    THEN InstLemma ‘length_nth_tl‘ [ℕn;L;j+1] THENA Auto’
| . . . | |
| . . . | 1. n : ℕ
| . . . | 2. R : ℕn→ℕn→Prop
| . . . | 3. x : ℕn
| . . . | 4. y : ℕn
| . . . | 5. n@0 : ℕ
| . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | 7. L : ℕn List
| . . . | 8. ||L|| = n@0+1
| . . . | 9. L[0] = x
| . . . | 10. last(L) = y
| . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | 12. ¬n@0≤n
```

```
| . . . | 13. i : ℕ(n@0+1)
| . . . | 14. j : ℕ(n@0+1)
| . . . | 15. i<j
| . . . | 16. L[i] = L[j]
| . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | 18. ||firstn(i+1;L)|| = i+1
| . . . | 19. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . | ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[(n@0-(j-i)+1-1)]
| . . . | |  =
| . . . | |  L[(||L||-1)]  by Decide j = n@0 THENA Auto
| . . . | |
| . . . | |\
| . . . | | 1. n : ℕ
| . . . | | 2. R : ℕn→ℕn→Prop
| . . . | | 3. x : ℕn
| . . . | | 4. y : ℕn
| . . . | | 5. n@0 : ℕ
| . . . | | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | | 7. L : ℕn List
| . . . | | 8. ||L|| = n@0+1
| . . . | | 9. L[0] = x
| . . . | | 10. last(L) = y
| . . . | | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | | 12. ¬n@0≤n
| . . . | | 13. i : ℕ(n@0+1)
| . . . | | 14. j : ℕ(n@0+1)
| . . . | | 15. i<j
| . . . | | 16. L[i] = L[j]
| . . . | | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | | 18. ||firstn(i+1;L)|| = i+1
| . . . | | 19. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . | | 20. j = n@0
| . . . | | ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[(n@0-(j-i)+1-1)]
| . . . | | |  =
| . . . | | |  L[(||L||-1)]
| . . . | | | by ...
| . . . | | |
| . . . | | 1. n : ℕ
| . . . | | 2. R : ℕn→ℕn→Prop
| . . . | | 3. x : ℕn
| . . . | | 4. y : ℕn
| . . . | | 5. n@0 : ℕ
| . . . | | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | | 7. L : ℕn List
| . . . | | 8. ||L|| = n@0+1
| . . . | | 9. L[0] = x
| . . . | | 10. last(L) = y
| . . . | | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | | 12. ¬n@0≤n
| . . . | | 13. i : ℕ(n@0+1)
| . . . | | 14. j : ℕ(n@0+1)
| . . . | | 15. i<j
| . . . | | 16. L[i] = L[j]
| . . . | | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | | 18. ||firstn(i+1;L)|| = i+1
```

```
| . . . | | 19. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . | | 20. j = n@0
| . . . | | ⊢  L[(n@0-(j-i)+1-1)] = L[(||L||-1)]
| . . . | |   by AssertBY n@0-(j-i)+1-1 = i
| . . . | |       Auto' THEN HypSubstSq (-1) 0
| . . . | | THEN AssertBY ||L||-1
| . . . | |                                                  =
| . . . | |                                                  j
| . . . | |       Auto' THEN HypSubstSq (-1) 0 THEN Auto
| . . . | |
| . . . | .\
| . . . | . 1. n : ℕ
| . . . | . 2. R : ℕn→ℕn→Prop
| . . . | . 3. x : ℕn
| . . . | . 4. y : ℕn
| . . . | . 5. n@0 : ℕ
| . . . | . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | . 7. L : ℕn List
| . . . | . 8. ||L|| = n@0+1
| . . . | . 9. L[0] = x
| . . . | . 10. last(L) = y
| . . . | . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | . 12. ¬n@0≤n
| . . . | . 13. i : ℕ(n@0+1)
| . . . | . 14. j : ℕ(n@0+1)
| . . . | . 15. i<j
| . . . | . 16. L[i] = L[j]
| . . . | . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | . 18. ||firstn(i+1;L)|| = i+1
| . . . | . 19. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . | . 20. ¬j = n@0
| . . . | . ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[(n@0-(j-i)+1-1)]
| . . . | . |  =
| . . . | . |  L[(||L||-1)]
| . . . | . | by ...
| . . . | . |
| . . . | . 1. n : ℕ
| . . . | . 2. R : ℕn→ℕn→Prop
| . . . | . 3. x : ℕn
| . . . | . 4. y : ℕn
| . . . | . 5. n@0 : ℕ
| . . . | . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . | . 7. L : ℕn List
| . . . | . 8. ||L|| = n@0+1
| . . . | . 9. L[0] = x
| . . . | . 10. last(L) = y
| . . . | . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . | . 12. ¬n@0≤n
| . . . | . 13. i : ℕ(n@0+1)
| . . . | . 14. j : ℕ(n@0+1)
| . . . | . 15. i<j
| . . . | . 16. L[i] = L[j]
| . . . | . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . | . 18. ||firstn(i+1;L)|| = i+1
| . . . | . 19. ||nth_tl(j+1;L)|| = ||L||-(j+1)
```

```
| . . . | . 20. ¬j = n@0
| . . . | . ⊢  L[(n@0-(j-i)+1-1-||firstn(i+1;L)||+j+1)] = L[(||L||-1)]
| . . . | .   by EqCD THEN Auto'
| . . . |
| . . . .\
| . . . . 1. n : ℕ
| . . . . 2. R : ℕn→ℕn→Prop
| . . . . 3. x : ℕn
| . . . . 4. y : ℕn
| . . . . 5. n@0 : ℕ
| . . . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . 7. L : ℕn List
| . . . . 8. ||L|| = n@0+1
| . . . . 9. L[0] = x
| . . . . 10. last(L) = y
| . . . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . 12. ¬n@0≤n
| . . . . 13. i : ℕ(n@0+1)
| . . . . 14. j : ℕ(n@0+1)
| . . . . 15. i<j
| . . . . 16. L[i] = L[j]
| . . . . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . 18. i@0 : ℕ(n@0-(j-i))
| . . . . ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[i@0]
| . . . . |  R (firstn(i+1;L) @ nth_tl(j+1;L))[(i@0+1)]
| . . . . | by InstLemma `length_firstn` [ℕn;L;i+1] THENA Auto'
| . . . .
| . . . . |    THEN InstLemma `length_nth_tl` [ℕn;L;j+1] THENA Auto'
| . . . . |
| . . . . 1. n : ℕ
| . . . . 2. R : ℕn→ℕn→Prop
| . . . . 3. x : ℕn
| . . . . 4. y : ℕn
| . . . . 5. n@0 : ℕ
| . . . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . 7. L : ℕn List
| . . . . 8. ||L|| = n@0+1
| . . . . 9. L[0] = x
| . . . . 10. last(L) = y
| . . . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . 12. ¬n@0≤n
| . . . . 13. i : ℕ(n@0+1)
| . . . . 14. j : ℕ(n@0+1)
| . . . . 15. i<j
| . . . . 16. L[i] = L[j]
| . . . . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . 18. i@0 : ℕ(n@0-(j-i))
| . . . . 19. ||firstn(i+1;L)|| = i+1
| . . . . 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . ⊢  (firstn(i+1;L) @ nth_tl(j+1;L))[i@0]
| . . . . |  R (firstn(i+1;L) @ nth_tl(j+1;L))[(i@0+1)]
| . . . . | by ...
| . . . .
| . . . . |    ...
| . . . . |
```

```
| . . . . |\
| . . . . | 1. n : ℕ
| . . . . | 2. R : ℕn→ℕn→Prop
| . . . . | 3. x : ℕn
| . . . . | 4. y : ℕn
| . . . . | 5. n@0 : ℕ
| . . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . | 7. L : ℕn List
| . . . . | 8. ||L|| = n@0+1
| . . . . | 9. L[0] = x
| . . . . | 10. last(L) = y
| . . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . | 12. ¬n@0≤n
| . . . . | 13. i : ℕ(n@0+1)
| . . . . | 14. j : ℕ(n@0+1)
| . . . . | 15. i<j
| . . . . | 16. L[i] = L[j]
| . . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . | 18. i@0 : ℕ(n@0-(j-i))
| . . . . | 19. ||firstn(i+1;L)|| = i+1
| . . . . | 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . | 21. i@0<||firstn(i+1;L)||
| . . . . | 22. i@0+1<||firstn(i+1;L)||
| . . . . | ⊢  L[i@0] R L[(i@0+1)]  by BackThruSomeHyp THEN Auto'
| . . . . |
| . . . . |\
| . . . . | 1. n : ℕ
| . . . . | 2. R : ℕn→ℕn→Prop
| . . . . | 3. x : ℕn
| . . . . | 4. y : ℕn
| . . . . | 5. n@0 : ℕ
| . . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . | 7. L : ℕn List
| . . . . | 8. ||L|| = n@0+1
| . . . . | 9. L[0] = x
| . . . . | 10. last(L) = y
| . . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . | 12. ¬n@0≤n
| . . . . | 13. i : ℕ(n@0+1)
| . . . . | 14. j : ℕ(n@0+1)
| . . . . | 15. i<j
| . . . . | 16. L[i] = L[j]
| . . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . | 18. i@0 : ℕ(n@0-(j-i))
| . . . . | 19. ||firstn(i+1;L)|| = i+1
| . . . . | 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . | 21. i@0<||firstn(i+1;L)||
| . . . . | 22. ||firstn(i+1;L)||≤i@0+1
| . . . . | ⊢  L[i@0] R L[(i@0+1-||firstn(i+1;L)||+j+1)]
| . . . . | | by AssertBY i@0 = i Auto' THEN HypSubstSq (-1) 0
| . . . . |
| . . . . | |    THEN AssertBY i+1-||firstn(i+1;L)||+j+1 = j+1 Auto'
| . . . . | |
| . . . . | THEN HypSubstSq (-1) 0
| . . . . | |
```

```
| . . . . | 1. n : ℕ
| . . . . | 2. R : ℕn→ℕn→Prop
| . . . . | 3. x : ℕn
| . . . . | 4. y : ℕn
| . . . . | 5. n@0 : ℕ
| . . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . | 7. L : ℕn List
| . . . . | 8. ||L|| = n@0+1
| . . . . | 9. L[0] = x
| . . . . | 10. last(L) = y
| . . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . | 12. ¬n@0≤n
| . . . . | 13. i : ℕ(n@0+1)
| . . . . | 14. j : ℕ(n@0+1)
| . . . . | 15. i<j
| . . . . | 16. L[i] = L[j]
| . . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . | 18. i@0 : ℕ(n@0-(j-i))
| . . . . | 19. ||firstn(i+1;L)|| = i+1
| . . . . | 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . | 21. i@0<||firstn(i+1;L)||
| . . . . | 22. ||firstn(i+1;L)||≤i@0+1
| . . . . | 23. i@0 = i
| . . . . | 24. i+1-||firstn(i+1;L)||+j+1 = j+1
| . . . . | ⊢  L[i] R L[(j+1)]  by WeakSubstFor L[i] 0 THENA Auto'
| . . . . | |
| . . . . | 1. n : ℕ
| . . . . | 2. R : ℕn→ℕn→Prop
| . . . . | 3. x : ℕn
| . . . . | 4. y : ℕn
| . . . . | 5. n@0 : ℕ
| . . . . | 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . | 7. L : ℕn List
| . . . . | 8. ||L|| = n@0+1
| . . . . | 9. L[0] = x
| . . . . | 10. last(L) = y
| . . . . | 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . | 12. ¬n@0≤n
| . . . . | 13. i : ℕ(n@0+1)
| . . . . | 14. j : ℕ(n@0+1)
| . . . . | 15. i<j
| . . . . | 16. L[i] = L[j]
| . . . . | 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . | 18. i@0 : ℕ(n@0-(j-i))
| . . . . | 19. ||firstn(i+1;L)|| = i+1
| . . . . | 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . | 21. i@0<||firstn(i+1;L)||
| . . . . | 22. ||firstn(i+1;L)||≤i@0+1
| . . . . | 23. i@0 = i
| . . . . | 24. i+1-||firstn(i+1;L)||+j+1 = j+1
| . . . . | ⊢  L[j] R L[(j+1)]  by BackThruSomeHyp THEN Auto'
| . . . . |
| . . . . .\
| . . . . . 1. n : ℕ
| . . . . . 2. R : ℕn→ℕn→Prop
```

```
| . . . . . 3. x : ℕn
| . . . . . 4. y : ℕn
| . . . . . 5. n@0 : ℕ
| . . . . . 6. ∀j:ℕ. j<n@0 ⇒ (x R^j y) ⇒ (∃k:ℕ(n+1). x R^k y)
| . . . . . 7. L : ℕn List
| . . . . . 8. ||L|| = n@0+1
| . . . . . 9. L[0] = x
| . . . . . 10. last(L) = y
| . . . . . 11. ∀i:ℕn@0. L[i] R L[(i+1)]
| . . . . . 12. ¬n@0≤n
| . . . . . 13. i : ℕ(n@0+1)
| . . . . . 14. j : ℕ(n@0+1)
| . . . . . 15. i<j
| . . . . . 16. L[i] = L[j]
| . . . . . 17. ||firstn(i+1;L) @ nth_tl(j+1;L)|| = n@0-(j-i)+1
| . . . . . 18. i@0 : ℕ(n@0-(j-i))
| . . . . . 19. ||firstn(i+1;L)|| = i+1
| . . . . . 20. ||nth_tl(j+1;L)|| = ||L||-(j+1)
| . . . . . 21. ||firstn(i+1;L)||≤i@0
| . . . . . 22. ||firstn(i+1;L)||≤i@0+1
| . . . . . ⊢  L[(i@0-||firstn(i+1;L)||+j+1)]
| . . . . .     R L[(i@0+1-||firstn(i+1;L)||+j+1)]
| . . . . .    by AssertBY i@0+1-||firstn(i+1;L)||+j+1
| . . . . .                =
| . . . . .                i@0-||firstn(i+1;L)||+j+1+1
| . . . . .
| . . . . .       ...
|
.\
. 1. n : ℕ
. 2. R : ℕn→ℕn→Prop
. 3. x : ℕn

. 4. y : ℕn
. 5. ∃k:ℕ(n+1). x R^k y
. ⊢  x (R^*) y
.   by ...
END
```