```
quotient_of_nsubn  THM

⊢  ∀n:{1...}, E:(ℕn→ℕn→Prop).
|  (EquivRel x,y:ℕn. x E y) & (∀x,y:ℕn. Dec(x E y)) ⇒
|  (∃m:ℕ(n+1). ℕm ∼ (i,j:ℕn//(i E j)))
| by BackThru
|    Thm* ∀i:ℤ, E:({i...}→Prop).
|         E(i) ⇒ (∀k:{i+1...}. E(k-1) ⇒ E(k)) ⇒ (∀k:{i...}. E(k))
|    {int_upper_ind}
|
|\
| ⊢  ∀E:(ℕ1→ℕ1→Prop).
| |  (EquivRel x,y:ℕ1. x E y) & (∀x,y:ℕ1. Dec(x E y)) ⇒
| |  (∃m:ℕ(1+1). ℕm ∼ (i,j:ℕ1//(i E j)))  by UnivCD THEN ExRepD
| |
| 1. E : ℕ1→ℕ1→Prop
| 2. EquivRel x,y:ℕ1. x E y
| 3. ∀x,y:ℕ1. Dec(x E y)
| ⊢  ∃m:ℕ(1+1). ℕm ∼ (i,j:ℕ1//(i E j))  by InstConcl [1]
| |
| ⊢  ℕ1 ∼ (i,j:ℕ1//(i E j))  by Unfold ‘one_one_corr‘ 0
| |
| ⊢  ∃f:(ℕ1→(i,j:ℕ1//(i E j))), g:((i,j:ℕ1//(i E j))→ℕ1).
| |  InvFuns(ℕ1; i,j:ℕ1//(i E j); f; g)
| | by Assert (0 ∈ i,j:ℕ1//(i E j))
| |
| |\
| | .....assertion .....
| | ⊢  0 ∈ i,j:ℕ1//(i E j)  by Assert (0 ∈ ℕ1)
| | |
| | 4. 0 ∈ ℕ1
| | ⊢   by Inclusion 4
| |
| .\
| . 4. 0 ∈ i,j:ℕ1//(i E j)
| . ⊢   by InstConcl [λx.0;λx.0]
| . |
| . ⊢  InvFuns(ℕ1; i,j:ℕ1//(i E j); λx.0; λx.0)
| . | by (Unfold ‘inv_funs‘ 0) THEN (Analyze 0)
| . |
| . |\
| . | ⊢  (λx.0) o (λx.0) = Id
| . |   by Eval [‘compose‘;‘tidentity‘;‘identity‘] 0 THEN Analyze
| . |
| . .\
| . . ⊢  (λx.0) o (λx.0) = Id
| . . | by Eval [‘compose‘;‘tidentity‘;‘identity‘] 0 THEN Analyze
| . . |
| . . 5. x : i,j:ℕ1//(i E j)
| . . ⊢  0 = x  by (Analyze 5) THEN Analyze
| . . |
| . . 5. x1 : ℕ1
| . . 6. x2 : ℕ1
| . . 7. x1 E x2
| . . ⊢  0 E x2  by Assert (x2 = 0)
```

```
| . . |
| . . 8. x2 = 0
| . . ⊢   by HypSubst 8 0
| . . |
| . . ⊢  0 E 0  by Analyze 2
| . . |
| . . 2. Refl(ℕ1;x,y.x E y)
| . . 3. Sym x,y:ℕ1. x E y & Trans x,y:ℕ1. x E y
| . . 4. ∀x,y:ℕ1. Dec(x E y)
| . . 5. 0 ∈ i,j:ℕ1//(i E j)
| . . 6. x1 : ℕ1
| . . 7. x2 : ℕ1
| . . 8. x1 E x2
| . . 9. x2 = 0
| . . ⊢   by Unfold ‘refl‘ 2
| . . |
| . . 2. ∀a:ℕ1. a E a
| . . 3. Sym x,y:ℕ1. x E y & Trans x,y:ℕ1. x E y
| . . 4. ∀x,y:ℕ1. Dec(x E y)
| . . 5. 0 ∈ i,j:ℕ1//(i E j)
| . . 6. x1 : ℕ1
| . . 7. x2 : ℕ1
| . . 8. x1 E x2
| . . 9. x2 = 0
| . . ⊢   by BackThru 2
|
.\
. ⊢  ∀n:{1+1...}.
. |  (∀E:(ℕ(n-1)→ℕ(n-1)→Prop).
. |   (EquivRel x,y:ℕ(n-1). x E y) & (∀x,y:ℕ(n-1). Dec(x E y)) ⇒
. |   (∃m:ℕ(n-1+1). ℕm ∼ (i,j:ℕ(n-1)//(i E j))))
. |  ⇒
. |
. |  (∀E:(ℕn→ℕn→Prop).
. |   (EquivRel x,y:ℕn. x E y) & (∀x,y:ℕn. Dec(x E y)) ⇒
. |   (∃m:ℕ(n+1). ℕm ∼ (i,j:ℕn//(i E j))))  by UnivCD
. |
. 1. n : {1+1...}
. 2. ∀E:(ℕ(n-1)→ℕ(n-1)→Prop).
. |  (EquivRel x,y:ℕ(n-1). x E y) & (∀x,y:ℕ(n-1). Dec(x E y)) ⇒
. |  (∃m:ℕ(n-1+1). ℕm ∼ (i,j:ℕ(n-1)//(i E j)))
. 3. E : ℕn→ℕn→Prop
. 4. (EquivRel x,y:ℕn. x E y) & (∀x,y:ℕn. Dec(x E y))
. ⊢  ∃m:ℕ(n+1). ℕm ∼ (i,j:ℕn//(i E j))
. | by Assert (EquivRel x,y:ℕ(n-1). x E y)
. |
. |\
. | 4. EquivRel x,y:ℕn. x E y
. | 5. ∀x,y:ℕn. Dec(x E y)
. | ⊢  EquivRel x,y:ℕ(n-1). x E y
. |   by BackThru
. |      Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. |           (EquivRel x,y:ℕn. x E y) ⇒ (EquivRel x,y:ℕm. x E y)
. |      {rest_equi_rel}
. |
```

```
. .\
. . 5. EquivRel x,y:ℕ(n-1). x E y
. . ⊢   by (InstHyp [E] 2) THEN ExRepD
. . |
. . 4. EquivRel x,y:ℕn. x E y
. . 5. ∀x,y:ℕn. Dec(x E y)
. . 6. EquivRel x,y:ℕ(n-1). x E y
. . 7. m : ℕ(n-1+1)
. . 8. ℕm ∼ (i,j:ℕ(n-1)//(i E j))
. . ⊢   by (Unfold ‘one_one_corr‘ 8) THEN ExRepD
. . |
. . 8. f : ℕm→(i,j:ℕ(n-1)//(i E j))
. . 9. g : (i,j:ℕ(n-1)//(i E j))→ℕm
. . 10. InvFuns(ℕm; i,j:ℕ(n-1)//(i E j); f; g)
. . ⊢   by Assert (∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j))
. . |
. . |\
. . | 11. x : ℕm
. . | ⊢  f(x) ∈ i,j:ℕn//(i E j)
. . |   by BackThru
. . |      Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . |           (EquivRel x,y:ℕn. x E y) ⇒
. . |           (∀z:x,y:ℕm//(x E y). z ∈ x,y:ℕn//(x E y))
. . |      {incl_aux2_quo}
. . |
. . .\
. . . 11. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
. . . ⊢   by (CopyToEnd 4) THEN (Unfold ‘equiv_rel‘ 12) THEN ExRepD
. . . |
. . . 12. Refl(ℕn;x,y.x E y)
. . . 13. Sym x,y:ℕn. x E y
. . . 14. Trans x,y:ℕn. x E y
. . . ⊢
. . . | by (Unfold ‘refl‘ 12) THEN (Unfold ‘sym‘ 13)
. . . |    THEN
. . . |    (Unfold ‘trans‘ 14)
. . . |    THEN
. . . |    (Unfold ‘one_one_corr‘ 0)
. . . |
. . . 12. ∀a:ℕn. a E a
. . . 13. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . 14. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . ⊢  ∃m:ℕ(n+1), f:(ℕm→(i,j:ℕn//(i E j)))
. . . |  , g:((i,j:ℕn//(i E j))→ℕm).
. . . |  InvFuns(ℕm; i,j:ℕn//(i E j); f; g)
. . . | by Assert (∀x,y:i,j:ℕn//(i E j). Dec(x = y))
. . . |
. . . |\
. . . | .....assertion .....
. . . | ⊢  ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . |   by RepD
. . . |      THEN
. . . |      (BackThru
. . . |       Thm* ∀E:(T→T→Prop).
. . . |            (EquivRel x,y:T. E(x,y)) ⇒
```

```
. . . |            (∀x,y:T. Dec(E(x,y))) ⇒
. . . |            (∀u,v:x,y:T//E(x,y). Dec(u = v))
. . . |       {decidable__quotient_equal})
. . . |
. . . .\
. . . . 15. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . ⊢
. . . . | by FwdThru
. . . . |    Thm* ∀E:(T→T→Prop).
. . . . |         (∀x,y:T. Dec(E(x,y)))
. . . . |         ⟺
. . . . |         (∃f:(T→T→𝔹). ∀x,y:T. (x f y) ⟺ E(x,y))
. . . . |    {dec_iff_ex_bvfun}
. . . . |    [15]
. . . . |
. . . . 16. ∃f:((i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹).
. . . . |   ∀x,y:i,j:ℕn//(i E j). (x f y) ⟺ x = y
. . . . ⊢   by New [‘Eb’] (Analyze 16)
. . . . |
. . . . 16. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . 17. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . ⊢   by Decide (∃k:ℕ(n-1). k E (n-1))
. . . . |
. . . . |\
. . . . | 18. ∃k:ℕ(n-1). k E (n-1)
. . . . | ⊢   by Analyze 18
. . . . | |
. . . . | 18. k : ℕ(n-1)
. . . . | 19. k E (n-1)
. . . . | ⊢
. . . . | | by InstConcl
. . . . | |    [m;λx.f(x);λx.if x Eb (n-1)→ g(k) else g(x) fi]
. . . . | |
. . . . | |\
. . . . | | .....wf .....
. . . . | | ⊢  m ∈ ℕ(n+1)  by Auto
. . . . | |

. . . . | |\
. . . . | | .....wf .....
. . . . | | ⊢  (λx.f(x)) ∈ ℕm→(i,j:ℕn//(i E j))  by Auto
. . . . | | |
. . . . | | 20. x : ℕm
. . . . | | ⊢  f(x) ∈ i,j:ℕn//(i E j)  by InstHyp [x] 11
. . . . | |
. . . . | |\
. . . . | | .....wf .....
. . . . | | ⊢  (λx.if x Eb (n-1)→ g(k) else g(x) fi)
. . . . | | |  ∈ (i,j:ℕn//(i E j))→ℕm  by Analyze
. . . . | | |
. . . . | | 20. x : i,j:ℕn//(i E j)
. . . . | | ⊢  if x Eb (n-1)→ g(k) else g(x) fi ∈ ℕm
. . . . | | | by (Unfold ‘member‘ 0) THEN (Analyze 20)
. . . . | | |
. . . . | | 20. x1 : ℕn
. . . . | | 21. x2 : ℕn
```

```
. . . . | | 22. x1 E x2
. . . . | | ⊢  if x1 Eb (n-1)→ g(k) else g(x1) fi
. . . . | | |  =
. . . . | | |  if x2 Eb (n-1)→ g(k) else g(x2) fi
. . . . | | | by Assert (n-1 ∈ i,j:ℕn//(i E j))
. . . . | | |
. . . . | | |\
. . . . | | | .....assertion .....
. . . . | | | ⊢  n-1 ∈ i,j:ℕn//(i E j)
. . . . | | |   by (Assert (n-1 ∈ ℕn)) THEN (Inclusion 23)
. . . . | | |
. . . . | | .\
. . . . | | . 23. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . ⊢   by SplitOnConclITEs
. . . . | | . |
. . . . | | . |\
. . . . | | . | .....truecase .....
. . . . | | . | 24. x1 Eb (n-1)
. . . . | | . | 25. x2 Eb (n-1)
. . . . | | . | ⊢  g(k) = g(k)  by Auto THEN Fold `member` 0
. . . . | | . |
. . . . | | . |\
. . . . | | . | .....falsecase .....
. . . . | | . | 24. x1 Eb (n-1)
. . . . | | . | 25. ¬(x2 Eb (n-1))
. . . . | | . | ⊢  g(k) = g(x2)  by Analyze
. . . . | | . | |
. . . . | | . | ⊢  k = x2  by Analyze
. . . . | | . | |
. . . . | | . | |\
. . . . | | . | | ⊢  x2<n-1  by Assert (¬x2 = n-1)
. . . . | | . | | |
. . . . | | . | | |\
. . . . | | . | | | .....assertion .....
. . . . | | . | | | ⊢  ¬x2 = n-1
. . . . | | . | | | | by (Analyze 0) THEN (Assert (x2 = n-1))
. . . . | | . | | | |
. . . . | | . | | | |\
. . . . | | . | | | | .....assertion .....
. . . . | | . | | | | 26. x2 = n-1
. . . . | | . | | | | ⊢  x2 = n-1  by Analyze
. . . . | | . | | | | |
. . . . | | . | | | | ⊢  x2 E (n-1)  by HypSubst 26 0
. . . . | | . | | | | |
. . . . | | . | | | | ⊢  (n-1) E (n-1)  by Analyze 4
. . . . | | . | | | | |
. . . . | | . | | | | 4. Refl(ℕn;x,y.x E y)
. . . . | | . | | | | 5. Sym x,y:ℕn. x E y & Trans x,y:ℕn. x E y
. . . . | | . | | | | 6. ∀x,y:ℕn. Dec(x E y)
. . . . | | . | | | | 7. EquivRel x,y:ℕ(n-1). x E y
. . . . | | . | | | | 8. m : ℕ(n-1+1)
. . . . | | . | | | | 9. f : ℕm→(i,j:ℕ(n-1)//(i E j))
. . . . | | . | | | | 10. g : (i,j:ℕ(n-1)//(i E j))→ℕm
. . . . | | . | | | | 11. InvFuns(ℕm; i,j:ℕ(n-1)//(i E j); f; g)
. . . . | | . | | | | 12. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
```

```
. . . . | | . | | | | 13. ∀a:ℕn. a E a
. . . . | | . | | | | 14. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . . | | . | | | | 15. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . . | | . | | | | 16. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . | | . | | | | 17. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . | | . | | | | 18. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . | | . | | | | 19. k : ℕ(n-1)
. . . . | | . | | | | 20. k E (n-1)
. . . . | | . | | | | 21. x1 : ℕn
. . . . | | . | | | | 22. x2 : ℕn
. . . . | | . | | | | 23. x1 E x2
. . . . | | . | | | | 24. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . | | | | 25. x1 Eb (n-1)
. . . . | | . | | | | 26. ¬(x2 Eb (n-1))
. . . . | | . | | | | 27. x2 = n-1
. . . . | | . | | | | ⊢   by Unfold `refl` 4
. . . . | | . | | | | |
. . . . | | . | | | | 4. ∀a:ℕn. a E a
. . . . | | . | | | | 5. Sym x,y:ℕn. x E y & Trans x,y:ℕn. x E y
. . . . | | . | | | | 6. ∀x,y:ℕn. Dec(x E y)
. . . . | | . | | | | 7. EquivRel x,y:ℕ(n-1). x E y
. . . . | | . | | | | 8. m : ℕ(n-1+1)
. . . . | | . | | | | 9. f : ℕm→(i,j:ℕ(n-1)//(i E j))
. . . . | | . | | | | 10. g : (i,j:ℕ(n-1)//(i E j))→ℕm
. . . . | | . | | | | 11. InvFuns(ℕm; i,j:ℕ(n-1)//(i E j); f; g)
. . . . | | . | | | | 12. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
. . . . | | . | | | | 13. ∀a:ℕn. a E a
. . . . | | . | | | | 14. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . . | | . | | | | 15. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . . | | . | | | | 16. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . | | . | | | | 17. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . | | . | | | | 18. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . | | . | | | | 19. k : ℕ(n-1)
. . . . | | . | | | | 20. k E (n-1)
. . . . | | . | | | | 21. x1 : ℕn
. . . . | | . | | | | 22. x2 : ℕn
. . . . | | . | | | | 23. x1 E x2
. . . . | | . | | | | 24. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . | | | | 25. x1 Eb (n-1)
. . . . | | . | | | | 26. ¬(x2 Eb (n-1))
. . . . | | . | | | | 27. x2 = n-1
. . . . | | . | | | | ⊢   by BackThru 4
. . . . | | . | | | |
. . . . | | . | | | .\
. . . . | | . | | | . 26. x2 = n-1
. . . . | | . | | | . 27. x2 = n-1
. . . . | | . | | | . ⊢  False  by InstHyp [x2;n-1] 17
. . . . | | . | | | . |
. . . . | | . | | | . 28. (x2 Eb (n-1)) ⟺ x2 = n-1
. . . . | | . | | | . ⊢   by Auto
. . . . | | . | | | . |
. . . . | | . | | | . 28. (x2 Eb (n-1)) ⇒ x2 = n-1
. . . . | | . | | | . 29. x2 Eb (n-1) ⇐ (x2 = n-1)
. . . . | | . | | | . ⊢   by Analyze 29
. . . . | | . | | |
```

```
. . . . | | . | | .\
. . . . | | . | | . 26. ¬x2 = n-1
. . . . | | . | | . ⊢   by SupInf
. . . . | | . | |
. . . . | | . | .\
. . . . | | . | . ⊢  k E x2  by Analyze 25
. . . . | | . | . |
. . . . | | . | . 24. x1 Eb (n-1)
. . . . | | . | . ⊢  x2 Eb (n-1)
. . . . | | . | . | by InstHyp [x1;n-1] 17 THEN InstHyp [x2;n-1] 17
. . . . | | . | . |
. . . . | | . | . 25. (x1 Eb (n-1)) ⟺ x1 = n-1
. . . . | | . | . 26. (x2 Eb (n-1)) ⟺ x2 = n-1
. . . . | | . | . ⊢   by Analyze 25
. . . . | | . | . |
. . . . | | . | . 25. (x1 Eb (n-1)) ⇒ x1 = n-1
. . . . | | . | . 26. x1 Eb (n-1) ⇐ (x1 = n-1)
. . . . | | . | . 27. (x2 Eb (n-1)) ⟺ x2 = n-1
. . . . | | . | . ⊢   by Analyze 25
. . . . | | . | . |
. . . . | | . | . 25. x1 Eb (n-1) ⇐ (x1 = n-1)
. . . . | | . | . 26. (x2 Eb (n-1)) ⟺ x2 = n-1
. . . . | | . | . 27. x1 = n-1
. . . . | | . | . ⊢   by Analyze 26
. . . . | | . | . |
. . . . | | . | . 26. (x2 Eb (n-1)) ⇒ x2 = n-1
. . . . | | . | . 27. x2 Eb (n-1) ⇐ (x2 = n-1)
. . . . | | . | . 28. x1 = n-1
. . . . | | . | . ⊢   by BackThru 27
. . . . | | . | . |
. . . . | | . | . ⊢  x2 = n-1  by Analyze
. . . . | | . | . |
. . . . | | . | . ⊢  x2 E (n-1)  by Analyze 28
. . . . | | . | . |
. . . . | | . | . 28. x1 E (n-1)  [not for witness]
. . . . | | . | . ⊢   by InstHyp [x2;x1;n-1] 14
. . . . | | . | . |
. . . . | | . | . |\
. . . . | | . | . | .....antecedent .....
. . . . | | . | . | ⊢  x2 E x1  by BackThru 13
. . . . | | . | . |
. . . . | | . | . |\
. . . . | | . | . | .....antecedent .....
. . . . | | . | . | ⊢  x1 E (n-1)  by Unhide
. . . . | | . | . |
. . . . | | . | . .\
. . . . | | . | . . 29. x2 E (n-1)
. . . . | | . | . . ⊢   by Auto
. . . . | | . |
. . . . | | . |\
. . . . | | . | .....truecase .....
. . . . | | . | 24. ¬(x1 Eb (n-1))
. . . . | | . | 25. x2 Eb (n-1)
. . . . | | . | ⊢  g(x1) = g(k)  by Analyze
. . . . | | . | |
```

```
. . . . | | . | ⊢  x1 = k  by Analyze
. . . . | | . | |
. . . . | | . | |\
. . . . | | . | | ⊢  x1<n-1  by Assert (¬x1 = n-1)
. . . . | | . | | |
. . . . | | . | | |\
. . . . | | . | | | .....assertion .....
. . . . | | . | | | ⊢  ¬x1 = n-1
. . . . | | . | | | | by (Analyze 0) THEN (Assert (x1 = n-1))
. . . . | | . | | | |
. . . . | | . | | | |\
. . . . | | . | | | | .....assertion .....
. . . . | | . | | | | 26. x1 = n-1
. . . . | | . | | | | ⊢  x1 = n-1  by Inclusion 26
. . . . | | . | | | |
. . . . | | . | | | .\
. . . . | | . | | | . 26. x1 = n-1
. . . . | | . | | | . 27. x1 = n-1
. . . . | | . | | | . ⊢  False  by FwdThru 17 [27]
. . . . | | . | | |
. . . . | | . | | .\
. . . . | | . | | . 26. ¬x1 = n-1
. . . . | | . | | . ⊢   by SupInf
. . . . | | . | |
. . . . | | . | .\
. . . . | | . | . ⊢  x1 E k
. . . . | | . | . | by (Analyze 24) THEN (FwdThru 17 [24])
. . . . | | . | . |    THEN
. . . . | | . | . |    (Assert (x2 = x1))
. . . . | | . | . |
. . . . | | . | . |\
. . . . | | . | . | .....assertion .....
. . . . | | . | . | 24. x2 Eb (n-1)
. . . . | | . | . | 25. x2 = n-1
. . . . | | . | . | ⊢  x2 = x1  by Analyze
. . . . | | . | . | |
. . . . | | . | . | ⊢  x2 E x1  by BackThru 13
. . . . | | . | . |
. . . . | | . | . .\
. . . . | | . | . . 24. x2 Eb (n-1)
. . . . | | . | . . 25. x2 = n-1
. . . . | | . | . . 26. x2 = x1
. . . . | | . | . . ⊢  x1 Eb (n-1)  by HypSubst 26 24
. . . . | | . |
. . . . | | . .\
. . . . | | . . .....falsecase .....
. . . . | | . . 24. ¬(x1 Eb (n-1))
. . . . | | . . 25. ¬(x2 Eb (n-1))
. . . . | | . . ⊢  g(x1) = g(x2)
. . . . | | . . | by (Assert (x1<n-1)) THEN (Assert (x2<n-1))
. . . . | | . . |    THEN
. . . . | | . . |    (Assert (x1 = x2))
. . . . | | . . |
. . . . | | . . |\
. . . . | | . . | .....assertion .....
```

```
. . . . | | . . | ⊢  x1<n-1  by Assert (¬x1 = n-1)
. . . . | | . . | |
. . . . | | . . | |\
. . . . | | . . | | .....assertion .....
. . . . | | . . | | ⊢  ¬x1 = n-1  by Analyze 0
. . . . | | . . | | |
. . . . | | . . | | 26. x1 = n-1
. . . . | | . . | | ⊢  False  by Assert (x1 = n-1)
. . . . | | . . | | |
. . . . | | . . | | |\
. . . . | | . . | | | .....assertion .....
. . . . | | . . | | | ⊢  x1 = n-1  by Inclusion 26
. . . . | | . . | | |
. . . . | | . . | | .\
. . . . | | . . | | . 27. x1 = n-1
. . . . | | . . | | . ⊢   by FwdThru 17 [27]
. . . . | | . . | |
. . . . | | . . | .\
. . . . | | . . | . 26. ¬x1 = n-1
. . . . | | . . | . ⊢   by SupInf
. . . . | | . . |
. . . . | | . . |\
. . . . | | . . | .....assertion .....
. . . . | | . . | 26. x1<n-1
. . . . | | . . | ⊢  x2<n-1  by Assert (¬x2 = n-1)
. . . . | | . . | |
. . . . | | . . | |\
. . . . | | . . | | .....assertion .....
. . . . | | . . | | ⊢  ¬x2 = n-1  by Analyze 0
. . . . | | . . | | |
. . . . | | . . | | 27. x2 = n-1
. . . . | | . . | | ⊢  False  by Assert (x2 = n-1)
. . . . | | . . | | |
. . . . | | . . | | |\
. . . . | | . . | | | .....assertion .....
. . . . | | . . | | | ⊢  x2 = n-1  by Inclusion 27
. . . . | | . . | | |
. . . . | | . . | | .\
. . . . | | . . | | . 28. x2 = n-1
. . . . | | . . | | . ⊢   by FwdThru 17 [28]
. . . . | | . . | |
. . . . | | . . | .\
. . . . | | . . | . 27. ¬x2 = n-1
. . . . | | . . | . ⊢   by SupInf
. . . . | | . . |
. . . . | | . . |\
. . . . | | . . | .....assertion .....
. . . . | | . . | 26. x1<n-1
. . . . | | . . | 27. x2<n-1
. . . . | | . . | ⊢  x1 = x2  by Analyze
. . . . | | . . |
. . . . | | . . .\
. . . . | | . . . 26. x1<n-1
. . . . | | . . . 27. x2<n-1
. . . . | | . . . 28. x1 = x2
```

```
. . . . | | . . . ⊢   by ApplyFunToHypEquands g 28
. . . . | |
. . . . | |\
. . . . | | ⊢  InvFuns(ℕm; i,j:ℕn//(i E j); λx.f(x); λx.if x Eb (n-1)→
. . . . | | |                                              g(k)
. . . . | | |                                           else g(x) fi)
. . . . | | | by (Analyze 0)
. . . . | | |    THEN
. . . . | | |    (Eval [‘compose‘;‘tidentity‘;‘identity‘] 0)
. . . . | | |
. . . . | | |\
. . . . | | | ⊢  (λx.if (f(x)) Eb (n-1)→ g(k) else g(f(x)) fi)
. . . . | | | |  =
. . . . | | | |  (λx.x)  by Analyze
. . . . | | | |
. . . . | | | 20. x : ℕm
. . . . | | | ⊢  if (f(x)) Eb (n-1)→ g(k) else g(f(x)) fi = x
. . . . | | | | by Assert (f(x) ∈ i,j:ℕn//(i E j))
. . . . | | | |
. . . . | | | |\
. . . . | | | | .....assertion .....
. . . . | | | | ⊢  f(x) ∈ i,j:ℕn//(i E j)  by BackThru 11
. . . . | | | |
. . . . | | | .\
. . . . | | | . 21. f(x) ∈ i,j:ℕn//(i E j)
. . . . | | | . ⊢   by Assert (n-1 ∈ i,j:ℕn//(i E j))
. . . . | | | . |
. . . . | | | . |\
. . . . | | | . | .....assertion .....
. . . . | | | . | ⊢  n-1 ∈ i,j:ℕn//(i E j)
. . . . | | | . |   by (Assert (n-1 ∈ ℕn)) THEN (Inclusion 22)
. . . . | | | . |
. . . . | | | . .\
. . . . | | | . . 22. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | | . . ⊢   by SplitOnConclITEs
. . . . | | | . . |
. . . . | | | . . |\
. . . . | | | . . | .....truecase .....
. . . . | | | . . | 23. (f(x)) Eb (n-1)
. . . . | | | . . | ⊢  g(k) = x  by FwdThru 17 [23]
. . . . | | | . . | |
. . . . | | | . . | 24. f(x) = n-1
. . . . | | | . . | ⊢   by Assert (k = n-1)
. . . . | | | . . | |
. . . . | | | . . | |\
. . . . | | | . . | | .....assertion .....
. . . . | | | . . | | ⊢  k = n-1  by Analyze
. . . . | | | . . | |
. . . . | | | . . | .\
. . . . | | | . . | . 25. k = n-1
. . . . | | | . . | . ⊢   by RevHypSubst 24 25
. . . . | | | . . | . |
. . . . | | | . . | . 25. k = f(x)
. . . . | | | . . | . ⊢   by Assert (k = f(x))
. . . . | | | . . | . |
```

```
. . . . | | | . . | . |\
. . . . | | | . . | . | .....assertion .....
. . . . | | | . . | . | ⊢  k = f(x)
. . . . | | | . . | . | | by Inst
. . . . | | | . . | . | |    Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . | | | . . | . | |         (EquivRel i,j:ℕn. i E j) ⇒
. . . . | | | . . | . | |         (∀x,y:i,j:ℕn//(i E j).
. . . . | | | . . | . | |          x = y ∈ i,j:ℕn//(i E j) ⟺ x = y)
. . . . | | | . . | . | |    {incl_aux3_quo}
. . . . | | | . . | . | |    [n;n-1;E;k;f(x)]
. . . . | | | . . | . | |
. . . . | | | . . | . | 26. k = f(x) ⟺ k = f(x)
. . . . | | | . . | . | ⊢   by Analyze 26 THEN Analyze 26
. . . . | | | . . | . |
. . . . | | | . . | . .\
. . . . | | | . . | . . 26. k = f(x)
. . . . | | | . . | . . ⊢   by HypSubst 26 0
. . . . | | | . . | . . |
. . . . | | | . . | . . ⊢  g(f(x)) = x
. . . . | | | . . | . .   by (Unfold ‘inv_funs‘ 10) THEN (Analyze 10)
. . . . | | | . . | . .      THEN
. . . . | | | . . | . .      (RWH add_composeC 0)
. . . . | | | . . | . .      THEN
. . . . | | | . . | . .      (RWW "10" 0)
. . . . | | | . . | . .      THEN
. . . . | | | . . | . .      (Eval [‘tidentity‘;‘identity‘] 0)
. . . . | | | . . |
. . . . | | | . . .\
. . . . | | | . . . .....falsecase .....
. . . . | | | . . . 23. ¬((f(x)) Eb (n-1))
. . . . | | | . . . ⊢  g(f(x)) = x
. . . . | | | . . .   by (Unfold ‘inv_funs‘ 10) THEN (Analyze 10)
. . . . | | | . . .      THEN
. . . . | | | . . .      (RWH add_composeC 0)
. . . . | | | . . .      THEN
. . . . | | | . . .      (RWW "10" 0)
. . . . | | | . . .      THEN
. . . . | | | . . .      (Eval [‘tidentity‘;‘identity‘] 0)
. . . . | | |
. . . . | | .\
. . . . | | . ⊢  (λx.f(if x Eb (n-1)→ g(k) else g(x) fi)) = (λx.x)
. . . . | | . | by Analyze
. . . . | | . |
. . . . | | . 20. x : i,j:ℕn//(i E j)
. . . . | | . ⊢  f(if x Eb (n-1)→ g(k) else g(x) fi) = x
. . . . | | . | by Assert (n-1 ∈ i,j:ℕn//(i E j))
. . . . | | . |
. . . . | | . |\
. . . . | | . | .....assertion .....
. . . . | | . | ⊢  n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . |   by (Assert (n-1 ∈ ℕn)) THEN (Inclusion 21)
. . . . | | . |

. . . . | | . .\
. . . . | | . . 21. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . . ⊢   by SplitOnConclITEs
```

```
. . . . | | . . |
. . . . | | . . |\
. . . . | | . . | .....truecase .....
. . . . | | . . | 22. x Eb (n-1)
. . . . | | . . | ⊢  f(g(k)) = x
. . . . | | . . | | by (Unfold ‘inv_funs‘ 10) THEN (Analyze 10)
. . . . | | . . | |    THEN
. . . . | | . . | |    (RWH add_composeC 0)
. . . . | | . . | |
. . . . | | . . | 10. g o f = Id
. . . . | | . . | 11. f o g = Id
. . . . | | . . | 12. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
. . . . | | . . | 13. ∀a:ℕn. a E a
. . . . | | . . | 14. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . . | | . . | 15. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . . | | . . | 16. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . | | . . | 17. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . | | . . | 18. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . | | . . | 19. k : ℕ(n-1)
. . . . | | . . | 20. k E (n-1)
. . . . | | . . | 21. x : i,j:ℕn//(i E j)
. . . . | | . . | 22. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . . | 23. x Eb (n-1)
. . . . | | . . | ⊢  (f o g)(k) = x  by Assert (x = k)
. . . . | | . . | |
. . . . | | . . | |\
. . . . | | . . | | .....assertion .....
. . . . | | . . | | ⊢  x = k  by FwdThru 18 [23]
. . . . | | . . | | |
. . . . | | . . | | 24. x = n-1
. . . . | | . . | | ⊢   by HypSubst 24 0
. . . . | | . . | | |
. . . . | | . . | | |\
. . . . | | . . | | | ⊢  n-1 = k  by Analyze THEN (FwdThru 14 [20])
. . . . | | . . | | |
. . . . | | . . | | .\
. . . . | | . . | | . 25. i,j:ℕn//(i E j)
. . . . | | . . | | . ⊢  k ∈ i,j:ℕn//(i E j)
. . . . | | . . | | .   by (Assert (k ∈ ℕn)) THEN (Inclusion 26)
. . . . | | . . | |
. . . . | | . . | .\
. . . . | | . . | . 24. x = k
. . . . | | . . | . ⊢   by HypSubst 24 0
. . . . | | . . | . |
. . . . | | . . | . |\
. . . . | | . . | . | ⊢  (f o g)(k) = k  by Assert ((f o g)(k) = k)
. . . . | | . . | . | |
. . . . | | . . | . | |\
. . . . | | . . | . | | .....assertion .....
. . . . | | . . | . | | ⊢  (f o g)(k) = k  by Assert (k = k)
. . . . | | . . | . | | |
. . . . | | . . | . | | 25. k = k
. . . . | | . . | . | | ⊢
. . . . | | . . | . | |   by (RWW "11" 0)
. . . . | | . . | . | |      THEN
```

```
. . . . | | . . | . | |      (Eval [‘tidentity‘;‘identity‘] 0)
. . . . | | . . | . | |
. . . . | | . . | . | .\
. . . . | | . . | . | . 25. (f o g)(k) = k
. . . . | | . . | . | . ⊢
. . . . | | . . | . | . | by Inst
. . . . | | . . | . | . |    Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . | | . . | . | . |         (EquivRel i,j:ℕn. i E j) ⇒
. . . . | | . . | . | . |         (∀x,y:i,j:ℕm//(i E j).
. . . . | | . . | . | . |          x = y ∈ i,j:ℕn//(i E j) ⟺ x = y)
. . . . | | . . | . | . |    {incl_aux3_quo}
. . . . | | . . | . | . |    [n;n-1;E;(f o g)(k);k]
. . . . | | . . | . | . |
. . . . | | . . | . | . 26. (f o g)(k) = k ⟺ (f o g)(k) = k
. . . . | | . . | . | . ⊢   by Analyze 26 THEN Analyze 27
. . . . | | . . | . |
. . . . | | . . | . .\
. . . . | | . . | . . 25. i,j:ℕn//(i E j)
. . . . | | . . | . . ⊢  (f o g)(k) ∈ i,j:ℕn//(i E j)
. . . . | | . . | . . | by Assert ((f o g)(k) ∈ i,j:ℕ(n-1)//(i E j))
. . . . | | . . | . . |
. . . . | | . . | . . 26. (f o g)(k) ∈ i,j:ℕ(n-1)//(i E j)
. . . . | | . . | . . ⊢
. . . . | | . . | . .   by BackThru
. . . . | | . . | . .      Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . | | . . | . .           (EquivRel x,y:ℕn. x E y) ⇒
. . . . | | . . | . .           (∀z:x,y:ℕm//(x E y).
. . . . | | . . | . .            z ∈ x,y:ℕn//(x E y))
. . . . | | . . | . .      {incl_aux2_quo}
. . . . | | . . . |
. . . . | | . . . .\
. . . . | | . . . . .....falsecase .....
. . . . | | . . . . 22. ¬(x Eb (n-1))
. . . . | | . . . . ⊢  f(g(x)) = x
. . . . | | . . . . | by (Unfold ‘inv_funs‘ 10) THEN (Analyze 10)
. . . . | | . . . . |    THEN
. . . . | | . . . . |    (RWH add_composeC 0)
. . . . | | . . . . |
. . . . | | . . . . 10. g o f = Id
. . . . | | . . . . 11. f o g = Id
. . . . | | . . . . 12. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
. . . . | | . . . . 13. ∀a:ℕn. a E a
. . . . | | . . . . 14. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . . | | . . . . 15. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . . | | . . . . 16. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . | | . . . . 17. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . | | . . . . 18. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . | | . . . . 19. k : ℕ(n-1)
. . . . | | . . . . 20. k E (n-1)
. . . . | | . . . . 21. x : i,j:ℕn//(i E j)
. . . . | | . . . . 22. n-1 ∈ i,j:ℕn//(i E j)
. . . . | | . . . . 23. ¬(x Eb (n-1))
. . . . | | . . . . ⊢  (f o g)(x) = x  by Assert ((f o g)(x) = x)
. . . . | | . . . . |
. . . . | | . . . . |\
```

```
. . . . | | . . . | .....assertion .....
. . . . | | . . . | ⊢  (f o g)(x) = x  by Assert (x = x)
. . . . | | . . . | |
. . . . | | . . . | |\
. . . . | | . . . | | .....assertion .....
. . . . | | . . . | | ⊢  x = x  by RWW "18" 23
. . . . | | . . . | | |
. . . . | | . . . | | 23. ¬x = n-1
. . . . | | . . . | | ⊢
. . . . | | . . . | |   by Fold ‘member‘ 0
. . . . | | . . . | |      THEN
. . . . | | . . . | |      BackThru
. . . . | | . . . | |      Thm* ∀n:{1...}, E:(ℕn→ℕn→Prop).
. . . . | | . . . | |           (EquivRel i,j:ℕn. i E j) ⇒
. . . . | | . . . | |           (∀x:i,j:ℕn//(i E j).
. . . . | | . . . | |            ¬x = n-1 ∈ i,j:ℕn//(i E j) ⇒
. . . . | | . . . | |            x ∈ i,j:ℕ(n-1)//(i E j))
. . . . | | . . . | |      {incl_aux4_quo}
. . . . | | . . . | |
. . . . | | . . . | .\
. . . . | | . . . | . 24. x = x
. . . . | | . . . | . ⊢
. . . . | | . . . | .   by (RWW "11" 0)
. . . . | | . . . | .      THEN
. . . . | | . . . | .      (Eval [‘tidentity‘;‘identity‘] 0)
. . . . | | . . . |
. . . . | | . . . .\
. . . . | | . . . . 24. (f o g)(x) = x
. . . . | | . . . . ⊢
. . . . | | . . . . | by Inst
. . . . | | . . . . |    Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . | | . . . . |         (EquivRel i,j:ℕn. i E j) ⇒
. . . . | | . . . . |         (∀x,y:i,j:ℕm//(i E j).
. . . . | | . . . . |          x = y ∈ i,j:ℕn//(i E j) ⟺ x = y)
. . . . | | . . . . |    {incl_aux3_quo}
. . . . | | . . . . |    [n;n-1;E;(f o g)(x);x]
. . . . | | . . . . |
. . . . | | . . . . 25. (f o g)(x) = x ⟺ (f o g)(x) = x
. . . . | | . . . . ⊢   by Analyze 25 THEN Analyze 26
. . . . | |
. . . . | |\
. . . . | | .....wf .....
. . . . | | 20. g1 : (i,j:ℕn//(i E j))→ℕm
. . . . | | ⊢  InvFuns(ℕm; i,j:ℕn//(i E j); λx.f(x); g1) ∈ Prop
. . . . | | | by Auto
. . . . | | |
. . . . | | 21. x : ℕm
. . . . | | ⊢  f(x) ∈ i,j:ℕn//(i E j)  by BackThru 11
. . . . | |
. . . . | |\
. . . . | | .....wf .....
. . . . | | 20. f1 : ℕm→(i,j:ℕn//(i E j))
. . . . | | ⊢  (∃g:((i,j:ℕn//(i E j))→ℕm).
. . . . | |     InvFuns(ℕm; i,j:ℕn//(i E j); f1; g))
. . . . | |    ∈ Prop  by Auto
```

```
. . . . . | |
. . . . . | .\
. . . . . | . .....wf .....
. . . . . | . 20. m1 : ℕ(n+1)
. . . . . | . ⊢  (∃f:(ℕm1→(i,j:ℕn//(i E j)))
. . . . . | .     , g:((i,j:ℕn//(i E j))→ℕm1).
. . . . . | .     InvFuns(ℕm1; i,j:ℕn//(i E j); f; g))
. . . . . | .    ∈ Prop  by Auto
. . . . . |
. . . . . .\
. . . . . . 18. ¬(∃k:ℕ(n-1). k E (n-1))
. . . . . . ⊢   by Assert (n-1 ∈ i,j:ℕn//(i E j))
. . . . . . |
. . . . . . |\
. . . . . . | .....assertion .....
. . . . . . | ⊢  n-1 ∈ i,j:ℕn//(i E j)
. . . . . . |   by (Assert (n-1 ∈ ℕn)) THEN (Inclusion 19)
. . . . . . |
. . . . . . .\
. . . . . . . 19. n-1 ∈ i,j:ℕn//(i E j)
. . . . . . . ⊢   by Let (f1 = (λx.if x=₂m→ n-1 else f(x) fi))
. . . . . . . |
. . . . . . . |\
. . . . . . . | .....wf .....
. . . . . . . | ⊢  (λx.if x=₂m→ n-1 else f(x) fi)
. . . . . . . | |  ∈ ℕ(m+1)→(i,j:ℕn//(i E j))
. . . . . . . | | by Analyze THEN SplitOnConclITE
. . . . . . . | |
. . . . . . . | |\
. . . . . . . | | .....truecase .....
. . . . . . . | | 20. x : ℕ(m+1)
. . . . . . . | | 21. x = m
. . . . . . . | | ⊢  n-1 ∈ i,j:ℕn//(i E j)  by Auto
. . . . . . . | |
. . . . . . . | .\
. . . . . . . | . .....falsecase .....
. . . . . . . | . 20. x : ℕ(m+1)
. . . . . . . | . 21. ¬x = m
. . . . . . . | . ⊢  f(x) ∈ i,j:ℕn//(i E j)  by BackThru 11
. . . . . . . |
. . . . . . . |\
. . . . . . . | .....wf .....
. . . . . . . | ⊢  ℕ(m+1)→(i,j:ℕn//(i E j)) ∈ Type  by Auto
. . . . . . . |
. . . . . . . .\
. . . . . . . . 20. f1 : ℕ(m+1)→(i,j:ℕn//(i E j))
. . . . . . . . 21. f1 = (λx.if x=₂m→ n-1 else f(x) fi)
. . . . . . . . ⊢   by Let (g1 = (λx.if x Eb (n-1)→ m else g(x) fi))
. . . . . . . . |
. . . . . . . . |\
. . . . . . . . | .....wf .....
. . . . . . . . | ⊢  (λx.if x Eb (n-1)→ m else g(x) fi)
. . . . . . . . | |  ∈ (i,j:ℕn//(i E j))→ℕ(m+1)
. . . . . . . . | | by Analyze THEN SplitOnConclITE
. . . . . . . . | |
```

```
. . . . . . . . | |\
. . . . . . . . | | .....truecase .....
. . . . . . . . | | 22. x : i,j:ℕn//(i E j)
. . . . . . . . | | 23. x Eb (n-1)
. . . . . . . . | | ⊢  m ∈ ℕ(m+1)  by Auto
. . . . . . . . | |
. . . . . . . . | .\
. . . . . . . . | . .....falsecase .....
. . . . . . . . | . 22. x : i,j:ℕn//(i E j)
. . . . . . . . | . 23. ¬(x Eb (n-1))
. . . . . . . . | . ⊢  g(x) ∈ ℕ(m+1)
. . . . . . . . | .   by (RWO "17" 23)
. . . . . . . . | .      THEN
. . . . . . . . | .      (FwdThru
. . . . . . . . | .       Thm* ∀n:{1...}, E:(ℕn→ℕn→Prop).
. . . . . . . . | .            (EquivRel i,j:ℕn. i E j) ⇒
. . . . . . . . | .            (∀x:i,j:ℕn//(i E j).
. . . . . . . . | .             ¬x = n-1 ∈ i,j:ℕn//(i E j) ⇒
. . . . . . . . | .             x ∈ i,j:ℕ(n-1)//(i E j))
. . . . . . . . | .       {incl_aux4_quo}
. . . . . . . . | .       [23])
. . . . . . . . |
. . . . . . . . |\
. . . . . . . . | .....wf .....
. . . . . . . . | ⊢  (i,j:ℕn//(i E j))→ℕ(m+1) ∈ Type  by Auto
. . . . . . . . |
. . . . . . . . .\
. . . . . . . . . 22. g1 : (i,j:ℕn//(i E j))→ℕ(m+1)
. . . . . . . . . 23. g1 = (λx.if x Eb (n-1)→ m else g(x) fi)
. . . . . . . . . ⊢   by InstConcl [m+1;f1;g1]
. . . . . . . . . |
. . . . . . . . . ⊢  InvFuns(ℕ(m+1); i,j:ℕn//(i E j); f1; g1)
. . . . . . . . . | by Analyze 0
. . . . . . . . . |
. . . . . . . . . |\
. . . . . . . . . | ⊢  g1 o f1 = Id
. . . . . . . . . | | by Ext THEN (RWO "21 23" 0) THEN (Reduce 0)
. . . . . . . . . | |
. . . . . . . . . | 24. x : ℕ(m+1)
. . . . . . . . . | ⊢  if if x=₂m→ n-1 else f(x) fi Eb (n-1)→ m
. . . . . . . . . | |  else g(if x=₂m→ n-1 else f(x) fi) fi
. . . . . . . . . | |  =
. . . . . . . . . | |  x
. . . . . . . . . | | by ((SplitOnNthConclITE 2) THEN SplitOnConclITE)
. . . . . . . . . | |    THENA
. . . . . . . . . | |    Try (Complete Auto)
. . . . . . . . . | |
. . . . . . . . . | |\
. . . . . . . . . | | .....truecase .....
. . . . . . . . . | | 25. x = m
. . . . . . . . . | | 26. (n-1) Eb (n-1)
. . . . . . . . . | | ⊢  m = x  by Auto
. . . . . . . . . | |
. . . . . . . . . | |\
. . . . . . . . . | | .....falsecase .....
```

```
. . . . . . . . . | | 25. x = m
. . . . . . . . . | | 26. ¬((n-1) Eb (n-1))
. . . . . . . . . | | ⊢  g(n-1) = x  by Analyze 26 THEN BackThru 17
. . . . . . . . . | | |
. . . . . . . . . | | 25. x = m
. . . . . . . . . | | ⊢  n-1 = n-1  by Auto
. . . . . . . . . | |
. . . . . . . . . | |\
. . . . . . . . . | | .....wf .....
. . . . . . . . . | | 25. ¬x = m
. . . . . . . . . | | ⊢  ((f(x)) Eb (n-1)) ∈ 𝔹  by InstHyp [x] 11
. . . . . . . . . | |
. . . . . . . . . | |\
. . . . . . . . . | | .....truecase .....
. . . . . . . . . | | 25. ¬x = m
. . . . . . . . . | | 26. (f(x)) Eb (n-1)
. . . . . . . . . | | ⊢  m = x  by (FwdThru 17 [26]) THEN (Analyze 18)
. . . . . . . . . | | |
. . . . . . . . . | | |\
. . . . . . . . . | | | .....wf .....
. . . . . . . . . | | | ⊢  f(x) ∈ i,j:ℕn//(i E j)  by InstHyp [x] 11
. . . . . . . . . | | |

. . . . . . . . . | | .\
. . . . . . . . . | | . 18. n-1 ∈ i,j:ℕn//(i E j)
. . . . . . . . . | | . 19. f1 : ℕ(m+1)→(i,j:ℕn//(i E j))
. . . . . . . . . | | . 20. f1 = (λx.if x=₂m→ n-1 else f(x) fi)
. . . . . . . . . | | . 21. g1 : (i,j:ℕn//(i E j))→ℕ(m+1)
. . . . . . . . . | | . 22. g1 = (λx.if x Eb (n-1)→ m else g(x) fi)
. . . . . . . . . | | . 23. x : ℕ(m+1)
. . . . . . . . . | | . 24. ¬x = m
. . . . . . . . . | | . 25. (f(x)) Eb (n-1)
. . . . . . . . . | | . 26. f(x) = n-1
. . . . . . . . . | | . ⊢  ∃k:ℕ(n-1). k E (n-1)
. . . . . . . . . | | . | by (Let (y = f(x))) THEN (MoveToConcl 28)
. . . . . . . . . | | . |    THEN
. . . . . . . . . | | . |    CSquash
. . . . . . . . . | | . |    THEN
. . . . . . . . . | | . |    (UseEqWitness *)
. . . . . . . . . | | . |
. . . . . . . . . | | . 27. y : i,j:ℕ(n-1)//(i E j)
. . . . . . . . . | | . ⊢  * = *
. . . . . . . . . | | . | by (Analyze 27) THEN Analyze THEN GenRepD
. . . . . . . . . | | . |
. . . . . . . . . | | . 27. y1 : ℕ(n-1)
. . . . . . . . . | | . 28. y2 : ℕ(n-1)
. . . . . . . . . | | . 29. y1 E y2
. . . . . . . . . | | . 30. y1 = f(x)
. . . . . . . . . | | . ⊢  ∃k:ℕ(n-1). k E (n-1)  by Witness y1
. . . . . . . . . | | . |
. . . . . . . . . | | . ⊢  y1 E (n-1)
. . . . . . . . . | | . | by Inst
. . . . . . . . . | | . |    Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . . . . . . | | . |          (EquivRel i,j:ℕn. i E j) ⇒
. . . . . . . . . | | . |          (∀x,y:i,j:ℕn//(i E j).
. . . . . . . . . | | . |           x = y ∈ i,j:ℕn//(i E j) ⟺ x = y)
```

```
. . . . . . . . . | | . |    {incl_aux3_quo}
. . . . . . . . . | | . |    [n;n-1;E]
. . . . . . . . . | | . |
. . . . . . . . . | | . 31. ∀x,y:i,j:ℕ(n-1)//(i E j). x = y ⟺ x = y
. . . . . . . . . | | . ⊢   by RWH (RevHypC 31) 30
. . . . . . . . . | | . |
. . . . . . . . . | | . 30. y1 = f(x)
. . . . . . . . . | | . 31. ∀x,y:i,j:ℕ(n-1)//(i E j). x = y ⟺ x = y
. . . . . . . . . | | . ⊢   by Assert (y1 = n-1)
. . . . . . . . . | | . |
. . . . . . . . . | | . 32. y1 = n-1
. . . . . . . . . | | . ⊢   by Analyze 32 THEN Unhide
. . . . . . . . . | |
. . . . . . . . . | .\
. . . . . . . . . | . .....falsecase .....
. . . . . . . . . | . 25. ¬x = m
. . . . . . . . . | . 26. ¬((f(x)) Eb (n-1))
. . . . . . . . . | . ⊢  g(f(x)) = x
. . . . . . . . . | .   by Analyze 10 THEN RWH add_composeC 0
. . . . . . . . . | .      THEN
. . . . . . . . . | .      HypSubst 10 0
. . . . . . . . . | .      THEN
. . . . . . . . . | .      Reduce 0
. . . . . . . . . |
. . . . . . . . . .\
. . . . . . . . . . ⊢  f1 o g1 = Id
. . . . . . . . . . | by Ext THEN (RWO "21 23" 0) THEN (Reduce 0)
. . . . . . . . . . |    THEN
. . . . . . . . . . |    (SplitOnNthConclITE 2)
. . . . . . . . . . |
. . . . . . . . . . |\
. . . . . . . . . . | .....truecase .....
. . . . . . . . . . | 24. x : i,j:ℕn//(i E j)
. . . . . . . . . . | 25. x Eb (n-1)
. . . . . . . . . . | ⊢  if m=₂m→ n-1 else f(m) fi = x
. . . . . . . . . . | | by SplitOnConclITE
. . . . . . . . . . | |
. . . . . . . . . . | |\
. . . . . . . . . . | | .....truecase .....
. . . . . . . . . . | | 26. m = m
. . . . . . . . . . | | ⊢  n-1 = x  by SwapEquands 0
. . . . . . . . . . | | |
. . . . . . . . . . | | ⊢  x = n-1  by FwdThru 17 [25]
. . . . . . . . . . | |
. . . . . . . . . . | .\
. . . . . . . . . . | . .....falsecase .....
. . . . . . . . . . | . 26. ¬m = m
. . . . . . . . . . | . ⊢  f(m) = x  by Analyze 26
. . . . . . . . . . |
. . . . . . . . . . .\
. . . . . . . . . . . .....falsecase .....
. . . . . . . . . . . 24. x : i,j:ℕn//(i E j)
. . . . . . . . . . . 25. ¬(x Eb (n-1))
. . . . . . . . . . . ⊢  if g(x)=₂m→ n-1 else f(g(x)) fi = x
. . . . . . . . . . . | by (RWO "17" 25)
```

```
. . . . . . . . . . . . |    THEN
. . . . . . . . . . . . |    (FwdThru
. . . . . . . . . . . . |     Thm* ∀n:{1...}, E:(ℕn→ℕn→Prop).
. . . . . . . . . . . . |          (EquivRel i,j:ℕn. i E j) ⇒
. . . . . . . . . . . . |          (∀x:i,j:ℕn//(i E j).
. . . . . . . . . . . . |           ¬x = n-1 ∈ i,j:ℕn//(i E j) ⇒
. . . . . . . . . . . . |           x ∈ i,j:ℕ(n-1)//(i E j))
. . . . . . . . . . . . |     {incl_aux4_quo}
. . . . . . . . . . . . |     [25])
. . . . . . . . . . . . |
. . . . . . . . . . . . 25. ¬x = n-1
. . . . . . . . . . . . 26. x ∈ i,j:ℕ(n-1)//(i E j)
. . . . . . . . . . . . ⊢   by SplitOnConclITE
. . . . . . . . . . . . |
. . . . . . . . . . . . |\
. . . . . . . . . . . . | .....truecase .....
. . . . . . . . . . . . | 27. g(x) = m
. . . . . . . . . . . . | ⊢  n-1 = x  by Assert (g(x)<m)
. . . . . . . . . . . . | |
. . . . . . . . . . . . | |\
. . . . . . . . . . . . | | .....assertion .....
. . . . . . . . . . . . | | ⊢  g(x)<m  by SupInf'
. . . . . . . . . . . . | |
. . . . . . . . . . . . | .\
. . . . . . . . . . . . | . 28. g(x)<m
. . . . . . . . . . . . | . ⊢   by Auto
. . . . . . . . . . . . |
. . . . . . . . . . . . .\
. . . . . . . . . . . . . .....falsecase .....
. . . . . . . . . . . . . 27. ¬g(x) = m
. . . . . . . . . . . . . ⊢  f(g(x)) = x
. . . . . . . . . . . . . | by (Unfold `inv_funs` 10)
. . . . . . . . . . . . . |    THEN
. . . . . . . . . . . . . |    (RWH add_composeC 0)
. . . . . . . . . . . . . |
. . . . . . . . . . . . . 10. g o f = Id
. . . . . . . . . . . . . 11. f o g = Id
. . . . . . . . . . . . . 12. ∀x:ℕm. f(x) ∈ i,j:ℕn//(i E j)
. . . . . . . . . . . . . 13. ∀a:ℕn. a E a
. . . . . . . . . . . . . 14. ∀a,b:ℕn. (a E b) ⇒ (b E a)
. . . . . . . . . . . . . 15. ∀a,b,c:ℕn. (a E b) ⇒ (b E c) ⇒ (a E c)
. . . . . . . . . . . . . 16. ∀x,y:i,j:ℕn//(i E j). Dec(x = y)
. . . . . . . . . . . . . 17. Eb : (i,j:ℕn//(i E j))→(i,j:ℕn//(i E j))→𝔹
. . . . . . . . . . . . . 18. ∀x,y:i,j:ℕn//(i E j). (x Eb y) ⟺ x = y
. . . . . . . . . . . . . 19. ¬(∃k:ℕ(n-1). k E (n-1))
. . . . . . . . . . . . . 20. n-1 ∈ i,j:ℕn//(i E j)
. . . . . . . . . . . . . 21. f1 : ℕ(m+1)→(i,j:ℕn//(i E j))
. . . . . . . . . . . . . 22. f1 = (λx.if x=₂m→ n-1 else f(x) fi)
. . . . . . . . . . . . . 23. g1 : (i,j:ℕn//(i E j))→ℕ(m+1)
. . . . . . . . . . . . . 24. g1 = (λx.if x Eb (n-1)→ m else g(x) fi)
. . . . . . . . . . . . . 25. x : i,j:ℕn//(i E j)
. . . . . . . . . . . . . 26. ¬x = n-1
. . . . . . . . . . . . . 27. x ∈ i,j:ℕ(n-1)//(i E j)
. . . . . . . . . . . . . 28. ¬g(x) = m
. . . . . . . . . . . . . ⊢  (f o g)(x) = x
```

```
. . . . . . . . . . . . | by (HypSubst 11 0) THEN (Reduce 0)
. . . . . . . . . . . . |
. . . . . . . . . . . . 29. z :
. . . . . . . . . . . . |   (i,j:ℕ(n-1)//(i E j))→(i,j:ℕ(n-1)//(i E j))
. . . . . . . . . . . . ⊢  z(x) ∈ i,j:ℕn//(i E j)
. . . . . . . . . . . . | by Assert (z(x) ∈ i,j:ℕ(n-1)//(i E j))
. . . . . . . . . . . . |
. . . . . . . . . . . . 30. z(x) ∈ i,j:ℕ(n-1)//(i E j)
. . . . . . . . . . . . ⊢
. . . . . . . . . . . .   by Inst
. . . . . . . . . . . .      Thm* ∀n:{1...}, m:ℕn, E:(ℕn→ℕn→Prop).
. . . . . . . . . . . .           (EquivRel x,y:ℕn. x E y) ⇒
. . . . . . . . . . . .           (∀z:x,y:ℕm//(x E y).
. . . . . . . . . . . .            z ∈ x,y:ℕn//(x E y))
. . . . . . . . . . . .     {incl_aux2_quo}
. . . . . . . . . . . .     [n;n-1;E;z(x)]
END
```